MASTER'S DEGREE THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title: Production matrices and enumeration of geometric graphs

Author: Guillermo Esteban Pascual

Advisor: Clemens Huemer and Rodrigo I. Silveira

Department: Matemàtiques

Academic year: 2017-2018

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

Abstract

Enumeration of labelled geometric graphs is a famous problem in graph theory. A geometric graph on a set of points S is a graph with vertex set S whose edges are straight-line segments with endpoints in S. It is called non-crossing if no two edges intersect except at common endpoints. We focus on counting non-crossing geometric graphs on sets S of n points in convex position.

Despite there exist many methods for counting graphs, we focus on using generating trees and production matrices. Counting geometric graphs is then equivalent to calculate the powers of a production matrix. We use the latter using results on Riordan Arrays. Also, the idea of translating a combinatorial theory into a theory of infinite matrices is nowadays a current trend in discrete mathematics.

In this project we establish new formulas for the numbers of geometric graphs as well as combinatorial identities derived from the production matrices. A main contribution, compared with previous papers, is the different definition of vertex degree, which allows to obtain the novel production matrices.

We studied the following graph classes on point sets in convex position: k-angulations, geometric graphs, connected graphs, spanning trees and non-crossing partitions on n nodes. For each class, production matrices are obtained.

We then go on to study properties of these matrices that have combinatorial implications. In particular we obtain a vector that counts the number of graphs on n nodes with root vertex of degree j, the characteristic polynomial of such production matrices, eigenvectors of the matrices and other interesting properties.

Keywords:
geometric graph, production matrix, Riordan array

Contents

Chapter 1

Introduction

In combinatorics, graph enumeration describes a class of combinatorial enumeration problems in which one counts undirected or directed graphs of a certain class, typically as a function of the number of vertices of the graph. These problems may be solved either exactly (as an algebraic enumeration problem) or asymptotically. The pioneers in this area were Pólya [44], Cayley and Redfield [46].

In 1927 Redfield published a theorem that ten years later was rediscovered by George Pólya, who greatly popularized the result by applying it to many counting problems, in particular to the enumeration of chemical compounds. In some graphical enumeration problems, the vertices of the graph are considered to be labelled in such a way as to be distinguishable from each other, while in other problems any permutation of the vertices is considered to form the same graph, so the vertices are considered identical or unlabelled. In general, labelled problems tend to be easier, but thanks to the Pólya enumeration theorem we can reduce unlabelled problems to labelled ones: each unlabelled class is considered as a symmetry class of labelled objects.

The aim of this thesis is to work with the famous problem of enumerating labelled non-crossing geometric graphs on n vertices in convex position. A geometric graph can be defined as a graph with a vertex set $S = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^2$ of n points in convex position, whose edges are straight-line segments with endpoints in S. A graph is said to be non-crossing if no two edges of the graph intersect each other except at its common endpoints. Here, we only consider geometric graphs on sets of points in convex position, ordered counter-clockwise. Already in 1753, Euler and Segner determined the number of triangulations (i.e., non-crossing geometric graphs all whose faces except the exterior one are 3-gons). These numbers are the well-known Catalan numbers. For many other classes of non-crossing geometric graphs, such as trees, forests, dissections, non-crossing partitions, connected graphs and geometric graphs, Flajolet and Noy [18] gave formulas for their numbers. There are also other papers where such formulas were obtained, see for instance [34], where non-crossing partitions were discussed for the first time; [16], where trees were first enumerated or [9], where results for dissection were summarized.

One technique used for counting combinatorial objects are *production matrices.* In recent papers [28, 29, 30] on counting geometric graphs, production matrices have been derived for the following classes of graphs: triangulations, matchings, spanning trees, spanning forests, geometric graphs, and paths. In this work, we extend the previous results in several directions. We present new production matrices that count k-angulations, geometric graphs, connected geometric graphs, spanning trees, and non-crossing partitions and derive formulas for their numbers.

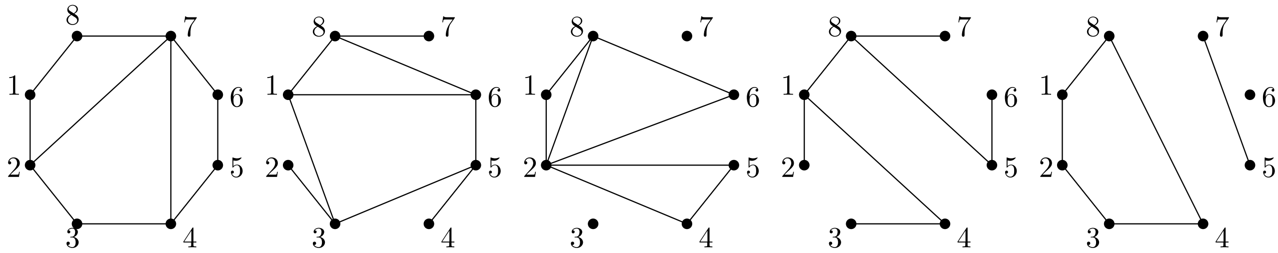

Figure 1.1: Figures that show the graph classes studied: 4-angulation, connected graph, geometric graph, spanning tree and non-crossing partition.

The key idea of production matrices is outlined in the following. Recently, in [28, 29, 30], it has been shown how geometric graphs can be counted by defining for each graph class an $n \times n$ production matrix A_n. The number of these graphs on a certain number of vertices are then given by (a column of) powers of A_n. To this end, we partition the graphs on $i \leq n$ vertices according to the *degree* of a specified root vertex. These production matrices A_n satisfy that $v^{i+1} = A_n v^i$, where the vector v^i counts the number of geometric graphs on i vertices. When we say the vector v^i counts the number of objects of a certain graph class on i vertices, we mean that each entry j of the vector v^i contains the number of graphs with root vertex of degree $j-1$, $\forall j = 1, \ldots, n$, and hence the sum of the elements gives the number of geometric graphs on i vertices. More precisely, the production matrix A_n then gives the relation $v^{i+1} = A_n v^i = A_n^{i+1-c} v^c$, when starting with a vector v^c for a constant number of vertices, which will usually be $(1, 0, ..., 0)^\top$ (for the graph class of geometric graphs the initial vector will be $(2, 0, ..., 0)^\top$). To find the production matrix A_n, the graphs are implicitly arranged in a tree structure, called *generating tree*, such that for each graph on i vertices and with root vertex of degree $j-1$, the number of its descendants on $i+1$ vertices with root vertex of degree ℓ (for each ℓ) is known.

For the graph class of k-angulations, the degree is defined as the number of edges incident to the root vertex, minus 2. We substract 2 because the two edges incident to the root vertex that are adjacent to the outer face are part of every k-angulation and hence they are disregarded. This argument is also applied to the degree of other graph classes. For geometric graphs, connected graphs and spanning trees, the degree of the root vertex is defined in a different way, based on visibility: the degree of the root vertex p_n is the number of vertices visible from a vertex p_{n+1} inserted between p_1 and p_n in convex position, minus 2. Two vertices are *visible* if the line segment connecting them does not intersect the interior of any edge of the graph. For non-crossing partitions, the root vertex degree is based on a diferent definition of visibility: the degree of the root vertex is the number of isolated vertices visible from a vertex p_{n+1} inserted between p_1 and p_n. Examples and explanations on how the production matrices are derived are given in the subsequent chapters.

Once the production matrix is obtained, we study its properties, such as its characteristic polynomial, its eigenvectors, and we also derived formulas for the vector v^i. To analyze the production matrices obtained we follow the approach of Merlini and Verri [38], using the theory of Riordan Arrays.

This chapter is devoted to introduce all the contents needed for the thesis. It includes both the main definitions that appear in other chapters, and the main theorems, relevant for this thesis, already existing in the literature. For some of the used results, however, we will only refer to the corresponding references. Also, some of the definitions, such as generating function, are not given because the reader can find them easily in the existing literature and we assume the reader is familiar with these concepts.

For the sake of clarity, the concepts explained in this section are divided into four sections. In Section 1.1, we define the analytic tools we will use in the following chapters. In Section 1.2 we state the main theorems that we will use. In Section 1.3 we show well-known results given in previous papers and their relation with some important sequences in mathematics, such as Fibonacci numbers or Catalan numbers. Finally, we conclude the chapter with a summary of the main results of the thesis.

1.1 Important tools

One of the tools used in this kind of enumerating problems are generating trees, a concept introduced in the literature by Chung et al. [7] to examine reduced Baxter permutations. This technique has been applied by West [60, 61] to other classes of permutations and more recently to some other combinatorial classes such as plane trees and lattice paths (see [3]). In all these cases, a generating tree is associated to a certain combinatorial class, according to some enumerative parameter, in such a way that the number of nodes appearing on level n of the tree gives the number of n-sized objects in the class.

Generating trees are the basis of the ECO method [3], and have been used to obtain matrix representations for combinatorial objects [14, 38]. For instance, the concept of generating tree was defined in [38] as follows:

Definition 1 ***(Generating tree)***

A generating tree is a rooted labelled tree with the property that if v_1 and v_2 are any two nodes with the same label, then for each label ℓ, v_1 and v_2 have exactly the same number of children with label ℓ. To specify a generating tree it therefore suffices to specify:

1. *the label of the root*

2. *a set of rules explaining how to derive from the label of a parent the labels of all of its children.*

For example, Figure 1.2 illustrates the first levels of the generating tree that corresponds to the following specification:

$$\begin{cases} \text{root: } (1) \\ \text{rules: } (k) \longrightarrow (1)(2)\cdots(k+1) \end{cases} \tag{1.1}$$

This rule is commonly denoted as *succession rule* [38].

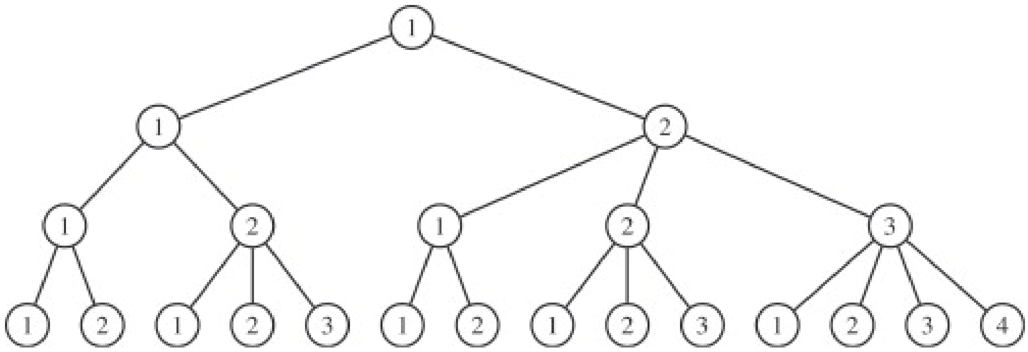

Figure 1.2: The partial generating tree for specification (1.1).

We can see the importance of generating trees when generating the parent of a given graph. We can easily obtain for each graph on n vertices, its parent on $n-1$ vertices. These rules will translate into the production matrices and guarantee that all the possible graphs of a certain graph class are produced exactly once.

A matrix derived from succession rules is also known as AGT matrix [38] (matrix associated to a generating tree).

Definition 2 ***(AGT matrix)***
An infinite matrix $\{d_{n,k}\}_{n,k\in\mathbb{N}}$ is said to be associated to a generating tree with root (c) (AGT matrix for short) if $d_{n,k}$ is the number of nodes at level n with label $k+c$. By convention, the level of the root is 0.

Refering to rule (1.1) and to Figure 1.2 we have the following partial associated matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & \cdots \\ 14 & 14 & 9 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.2}$$

where we recognize the Catalan triangle [10, 50, 55]. Each entry $a_{i,j}, \forall i \geq j$, of the matrix A is obtained as follow: $a_{i,j}$ is the number of nodes with label j at level $i-1$ in the generating tree.

One interesting quantity related to generating trees is the total number of nodes at level ℓ. By going to the AGT matrix associated to the generating tree, we see that this number is given by the row sums of the matrix. Thanks to the concept of generating trees, Barcucci et al. [3] describe a method for counting various combinatorial classes making use of it.

For graphs on point sets in convex position, generating trees for triangulations [31] and spanning trees [24] have already been obtained.

Some of the most remarkable theorems related to the generating tree concept are the following, which we will be using along this work and are explained in more detail in [38]. Proper Riordan Arrays are characterized by the following properties stated by Rogers [48] and studied in more detail by Sprugnoli [52].

Theorem 1 ***(Merlini et al. [38])***

A matrix $\{d_{n,k}\}_{n,k\in\mathbb{N}}$ is a proper Riordan Array iff there exists a sequence $A = \{a_i\}, i \in \mathbb{N}$ with $a_0 \neq 0$ such that every element $d_{n+1,k+1}$ can be expressed as a linear combination, with coefficients in A of the elements in the preceding row, starting from the preceding column:

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots$$

Theorem 2 ***(Merlini et al. [38])***

Let $\{d_{n,k}\}_{n,k\in\mathbb{N}}$ be a proper Riordan Array with $d_{n,n} \neq 0, \forall n \in \mathbb{N}$, then there exists a unique sequence $Z = \{z_i\}, i \in \mathbb{N}$ such that every element in column 0 can be expressed as a linear combination of all the elements in the preceding row:

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots$$

Thanks to these two theorems, we can define a proper Riordan Array [38] by the pair of generating functions $D = (d(t), h(t))$ where:

$$h(t) = A(th(t)) \qquad\qquad d(t) = \frac{d_0}{1 - tZ(th(t))} \tag{1.3}$$

with $d_0 = d(0)$.

The entries of the vector that counts the number of objects of a certain graph class are the values $d_{n,k}$, and therefore the sum of the entries of the vector v^i, which satisfies $v^i = Av^{i-1}$, where A is a production matrix, is the number of objects at level n. In other words, Riordan Arrays are a very useful tool to calculate the powers of matrices. So, one theorem that we will use especially when determining such entries of the vectors is the following.

Theorem 3 ***(Merlini et al. [38])***

Let $D = (d(t), h(t))$ be a proper Riordan Array, and let $A = \{a_j\}_{j\in\mathbb{N}}$ be its A-sequence. Then, if $d(t) = 1$ we have:

$$d_{n,k} = \frac{k}{n}[t^{n-k}]A(t)^n$$

and if $d(t) = h(t)$ we have:

$$d_{n,k} = \frac{k+1}{n+1}[t^{n-k}]A(t)^{n+1}$$

1.2 Matrix representation

As we follow the notation of Merlini and Verri [38], we consider working with matrix notation for our results. We will use many tools derived from matrix theory, such as the characteristic polynomial. Knowing the characteristic polynomial of a production matrix is interesting for several reasons. For instance, the following theorem then implies a relation among the number of graphs with given root vertex degree, as we will see in Section 3.1.3 for quadrangulations.

Theorem 4 ***(Cayley-Hamilton)***

Let $A \in M(n,n)$ where $M(n,n)$ is the set of all $n \times n$ matrices over a commutative ring with identity I_n and let

$$det(A - \lambda I_n) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0$$

be its characteristic polynomial. Then,

$$c_nA^n + c_{n-1}A^{n-1} + \cdots + c_0I_n = \mathbf{0}_n$$

This theorem, named after the mathematicians Arthur Cayley and William Rowan Hamilton, which states that every square matrix over a commutative ring (such as the real or complex field) satisfies its own characteristic equation, was first proved in 1853 [23] in terms of inverses of linear functions of quaternions, a non-commutative ring, by Hamilton. This corresponds to the special case of certain 4×4 real or 2×2 complex matrices. Then, Cayley in 1858 stated it for 3×3 and smaller matrices. The general case was first proved by Frobenius in 1878.

In our context, also the next well-known theorem is important in order to determine the behaviour of our production matrices when they are sufficiently large.

Theorem 5 ***(Perron-Frobenius)***

Let $A = \{a_{ij}\}_{i,j\in\mathbb{N}}$ be an $n \times n$ positive matrix: $a_{ij} > 0$ for $1 \leq i,j \leq n$. Then there is a positive real number r such that r is an eigenvalue of A and any other eigenvalue λ (possibly, complex) is strictly smaller than r in absolute value. Thus, the spectral radius $\rho(A)$ is equal to r.

In this thesis, the matrix entries are $a_{ij} \geq 0$ for $1 \leq i,j \leq n$, and a variant of this theorem also applies.

Theorem 5 implies that the exponential growth rate of the matrix powers A^k as k tends towards infinity is controlled by an eigenvalue of A with the largest absolute value. Thus, the largest eigenvalue of an $n \times n$ production matrix, when n tends towards infinity, gives the asymptotic growth of the number of graphs. So, another advantage of representing generating trees by a matrix is that we are able to estimate the asymptotic number of graphs of a certain class. We mention that, recently [15], extremal statistics in non-crossing configurations on the n vertices of a convex polygon have been analyzed. Flajolet and Noy [18] already determined the asympotitic number of many graph classes.

While the asymptotic value of the largest eigenvalue is known for some graph classes, the precise eigenvalues are known for only some production matrices such as the ones of the production matrix of triangulations by Chow in [6] and, as a corollary of this result, the eigenvalues of the production matrix for matchings have also been obtained [27].

1.3 Previous results

The starting point of this thesis are the papers of Huemer et al. [28, 29, 30]. In these papers, they give production matrices for some classes of graphs, and then, they obtain formulas for the characteristic polynomials of such matrices as well as formulas for the entries of the vectors that enumerate the number of graphs. So, in this section, we summarize some of the known results obtained in these papers of Huemer et al. For instance, in the table below, examples of the production matrices for different graph classes for $n = 6$ are presented. Matrix (f) is for paths on at most $\frac{n}{2}$ points.

$\begin{pmatrix} 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 0&1&1&1&1&1\\ 0&0&1&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&0&1&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&1&1&1&1\\ 1&0&1&1&1&1\\ 0&1&0&1&1&1\\ 0&0&1&0&1&1\\ 0&0&0&1&0&1\\ 0&0&0&0&1&0 \end{pmatrix}$	$\begin{pmatrix} 2&3&4&5&6&7\\ 1&2&3&4&5&6\\ 0&1&2&3&4&5\\ 0&0&1&2&3&4\\ 0&0&0&1&2&3\\ 0&0&0&0&1&2 \end{pmatrix}$
(a) Triangulations	(b) Matchings	(c) Spanning trees
$\begin{pmatrix} 1&1&1&1&1&1\\ 1&3&4&5&6&7\\ 0&1&3&4&5&6\\ 0&0&1&3&4&5\\ 0&0&0&1&3&4\\ 0&0&0&0&1&3 \end{pmatrix}$	$\begin{pmatrix} 1&1&1&1&1&1\\ 1&3&3&3&3&3\\ 0&2&4&4&4&4\\ 0&0&2&4&4&4\\ 0&0&0&2&4&4\\ 0&0&0&0&2&4 \end{pmatrix}$	$\begin{pmatrix} 1&1&1&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 1&1&1&1&1&1\\ 0&1&1&0&1&1\\ 0&0&1&0&0&1 \end{pmatrix}$
(d) Forests	(e) Geometric graphs	(f) Paths

Figure 1.3: Production matrices for six different graph classes.

In the next figure we present formulas for the k-th entry of the vector v^n from the paper [29]. These formulas give the number of graphs for each class on n vertices in which the root vertex has degree k.

Triangulations	$B_{n,k} = \frac{k+1}{n+1}\binom{2n-k}{n-k}$
Matchings	$\sum_{j=k-1}^{n} \binom{n}{j}(-1)^{n+j} B_{j,k-1}$
Spanning trees	$\frac{k}{n-1}\binom{3n-k-4}{n-k-1}$
Forests	$\frac{1}{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}\left(k\binom{n+2j-k-2}{j-k} + (k+1)\binom{n+2j-k-3}{j-k-1} - (k+2)\binom{n+2j-k-4}{j-k-2}\right)$
Geometric graphs	$\frac{2^{n-2}}{n-1}\sum_{j=0}^{n}\binom{n-1}{j}\left(k\binom{n+j-k-2}{j-k} - (k+2)\binom{n+j-k-4}{j-k-2}\right)$
Paths	$\lceil 2^{n-k-2} + (n-k)2^{n-k-3}\rceil$

Figure 1.4: Formulas for the number of graphs for each class with n vertices in which the root vertex has degree k. (The degree is defined in terms of visibility for matchings and paths). Some are only valid for $n > 1$ and $k > 0$. For paths, the formula corresponds to the sum of two entries of the vector.

Many of the obtained results are related to Catalan numbers [35, 54], Fibonacci numbers [33, 37], or Ballot numbers and these numbers appear several times in the formulas when calculating the characteristic polynomials or the number of graphs for some class. In the case of Catalan numbers, they count the number of triangulations, the number of perfect matchings or the number of non-crossing partitions. The Ballot numbers appear in the formula of the vector that counts the number of triangulations and also in the one for matchings due to its relation with Catalan numbers. Also, in [28] appears a relation between Fibonacci numbers and the class of graphs of forests.

In addition, for the characteristic polynomial we give a list of the results known for the same classes of graphs given in the paper of Huemer et al. [28]:

(a) triangulations	$t_n(\lambda) = \sum_{k=0}^{\lceil n/2 \rceil} \binom{n-k+1}{k} (-1)^{n+k} \lambda^{n-k}$
(b) matchings	$m_n(\lambda) = \sum_{k=0}^{\lceil n/2 \rceil} \sum_{j=0}^{n-k} \binom{n-k+1}{k} \binom{n-k}{j} (-1)^{n+k} \lambda^{j}$
(c) spanning trees	$s_n(\lambda) = \sum_{k=0}^{n} \binom{2k+2}{n-k} \lambda^k (-1)^k$
(d) forests	$f_n(\lambda) = \sum_{k=0}^{n} \lambda^k (-1)^k \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} \binom{\ell}{j} \binom{\ell}{2\ell+j-n} \binom{2\ell+j-n}{k+\ell+j-n}$
(e) geometric graphs	$g_n(\lambda) = 2^{n-1} + \sum_{k=1}^{n} \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{n-j}{k} 2^{n-k} (-1)^k \lambda^k$
(f) paths	$p_n(\lambda) = \sum_{k=0}^{n/2-1} \binom{n/2-1}{k} (-1)^{n/2+k+1} \lambda^k + \sum_{k=0}^{n/2} 2\frac{n-k}{n} \binom{n/2}{k} (-1)^{n/2+k} \lambda^{n/2+k}$

Figure 1.5: The characteristic polynomials of $n \times n$ production matrices for several graph classes.

1.4 Results

We will now explain the main contents of the thesis. In Chapter 2, we obtain two theorems for the so-called *Hessenberg-Toeplitz* matrices. These results are a recursive formula for the characteristic polynomial of the production matrices of the classes of graphs we study, and then we also obtain a formula for the entries of an eigenvector of such matrices, in terms of the characteristic polynomials. The eigenvectors are useful because we are able to find the eigenvalues associated to them applying numerical methods.

In Chapter 3, we work with the graph class of k-angulations. A k-angulation of a convex polygon is a 2-connected geometric graph where each interior face is a k-gon. The first step of our research is to study the base cases. For $k = 3$, the results are already known. For instance, Hurtado and Noy [31] obtain the generating tree for triangulations, Deutsch et al. [14] also give a production matrix T_n for triangulations, and for instance Huemer et al. [29] recall a well-known formula that allows to count the number of triangulations with point sets in convex position as well as a formula for the characteristic polynomial of the production matrix T_n. Hence, we start by studying the next cases of k-angulations for $k = 4$ and $k = 5$ and then, we propose a formula generalising all these previous results for any k. In these cases, we obtain all the formulas in terms of a parameter r, which represents the number of internal k-gons of the k-angulation, instead of using the number n of nodes. This is due to the fact that not for every number of nodes we can obtain a k-angulation.

The number of k-angulations of an n-gon is already known [47], but in this thesis we obtain all the results from a production matrix K_r as one can see in Section 3.3. These results are a formula for the entries of the vector v^r that enumerates the number of k-angulations in terms of the number r of k-gons (see Theorem 16). Each entry v_j^r of the vector represents the number of k-angulations with r k-gons and root vertex of degree $j-1$; in the case of k-angulations, the degree is defined as the number of incident edges to the root vertex $p_{(k-2)r+2}$, minus 2. We substract 2 to not consider the boundary edges of the convex polygon.

Also, we give a formula for the characteristic polynomial of the production matrix K_r as a recurrence relation (see Corollary 7) and its solution (see Theorem 17). Then we obtain a curious property for an eigenvector of the production matrix (see Corollary 8), relating the entries of it with the characteristic polynomials of smaller production matrices, evaluated at an eigenvalue. Other properties relating the characteristic polynomial and the vector of k-angulations have been obtained. For instance, Theorem 18 shows that the i-th row of the inverse of the Krylov matrix (explained in Section 3.3.4) of k-angulations corresponds to the $(ki+1)$-th row of the Pascal triangle with some entries multiplied by -1. This result gives a combinatorial identity (see Corollary 3).

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1 \\
\iddots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \ddots
\end{array}
$$

Figure 1.6: Pascal triangle

In Chapter 4 we study the class of all geometric graphs, or just graphs. Non-crossing geometric graphs have already been studied by Flajolet and Noy [18], giving a formula for the number of graphs of size n and k edges. The counting method used by Flajolet and Noy relies on generating functions, symbolic method, singularity analysis, and singularity perturbation. Huemer et al. [29] also work with this class of graphs. However, the method used in [29] is using the notion of generating trees and Riordan Arrays [38], so they give a production matrix G_n and then they compute the number of geometric graphs with root vertex of degree the number of incident edges incident to it. Knowing this production matrix, they also find a formula for the characteristic polynomial of G_n.

Since there are different ways of defining the degree of the root vertex, in this thesis we proceed to define a new production matrix for non-crossing geometric graphs. The difference between this new matrix (see Theorem 19) and the one that already existed (see Figure 4.1) is the way of defining the degree of the root vertex; we now define it in terms of visibility instead of using the number of adjacent vertices to the root vertex. We recall that a vertex p_i is visible from the root vertex p_n if we add a new vertex p_{n+1} in convex position between p_1 and p_n and the edge $p_{n+1}p_i$ does not cross any other edge of the graph.

In addition, we give a new formula for the vector that counts the number of geometric graphs different from one previously known (see Theorem 20), a recursive formula for the characteristic polynomial of the new production matrix (see Corollary 10) with its solution (see Theorem 21), and a formula for the entries of an eigenvector of the matrix (see Corollary 11). These entries have similar structure to the ones obtained for k-angulations, relating the entries of it with the characteristic polynomials of smaller production matrices, evaluated at an eigenvalue.

In Chapter 5, we work with the class of connected graphs. A graph is said to be connected if there is a path between every pair of vertices. In [18], Flajolet and Noy give a formula for the number of connected graphs on n vertices and another formula for the number of connected graphs on n vertices and k edges. However, in this thesis we propose a production matrix C_n that allows to count the number of connected graphs (see Theorem 22). This is the first time a production matrix for connected graphs has been obtained. Thanks to this production matrix, we state a formula for the entries of the vector that enumerates the number of connected graphs.

As we have said before, there are different ways of defining the degree of the root vertex, so in this case, we follow the same definition as for geometric graphs: visibility. In this case we also give a formula for the entries of the vector that counts the number of connected graphs (see Theorem 23) as well as a recursive formula for the characteristic polynomial of the productionmatrix C_n (see Corollary 12) and its solution (see Theorem 24). Hence, we propose a formula for the entries of an eigenvector of the production matrix C_n, evaluated at an eigenvalue (see Corollary 13). Once we know the number of connected graphs and the number of geometric graphs, we describe another production matrix that relates both graph classes. This matrix is proposed in Section 5.5 in Theorem 25, and its entries are given by the number of connected graphs and it gives the number of geometric graphs of the same graph class.

In Chapter 6, we work with spanning trees. A spanning tree is a connected graph that has no cycles. In this case, previous studies [29] give a production matrix S_n for this class of graphs defining the degree of the root vertex as the number of edges incident to it. And then, they find some properties such as the vector that gives the number of spanning trees with n nodes, or the characteristic polynomial of S_n [28].

A bijection between spanning trees and ternary trees [42] is also well known, as well as a bijection between ternary trees and quadrangulations, stated in Chapter 3, so the results of Chapter 6 about spanning trees are equal to the results we obtain in Section 3.1 about quadrangulations. However, we use a different definition for the degree of the root vertex from the one used with quadrangulations. We define the degree of the root vertex in terms of visibility. The result is that the production matrix obtained was equal to the one obtained for quadrangulations using another definition of the degree (see Theorem 28). Hence, the vector of spanning trees and the characteristic polynomial of the production matrix have the same formula as for quadrangulations.

Finally, Chapter 7 is devoted to non-crossing partitions. This class of graphs has been studied deeply because they are counted by Catalan numbers. Huemer et al. [29] propose a production matrix T_n defining the degree of the root vertex as the number of blocks visible from it. They also notice that there is a relation between non-crossing partitions and triangulations of point set in convex position. In [28], Huemer et al. calculate the corresponding vector for counting non-crossing partitions and the characteristic polynomial of the matrix T_n.

However, in this thesis we define the degree of the production matrix B_n of non-crossing partitions in a different way. Let ρ be a non-crossing partition with $\{p_1, \ldots, p_n\}$ points in convex position, then the degree of the root vertex is defined as the number of isolated vertices visible from a new node p_{n+1} inserted between p_1 and p_n. With this new root vertex degree definition we obtain a new production matrix (see Theorem 29). Then, we can obtain the number of non-crossing partitions (see Theorem 30) on n nodes and a given root vertex degree $j-1$. This formula is different from the one in [29]. We also obtain a recurrence relation for the characteristic polynomial and the solution for it (see Theorem 31), also different from the one obtained in [28].

We finally use a different approach which involves two production matrices. We study the number of non-crossing partitions for a fixed number of blocks, obtaining two production matrices F_n and H_n. The way we calculate the number of non-crossing partitions on $i+1$ vertices and $s+1$ blocks is summing up the two matrices F_n and H_n multiplied by $v^{i,s}$ and $v^{i,s+1}$ respectively, where $v^{i,s}$ is the vector that enumerates the number of non-crossing partitions with i vertices and s blocks. To obtain these matrices, we apply another definition of the root vertex: the number of vertices in the block of the root vertex p_n.

We finish the thesis with the conclusions of the work. We summarize the results we obtain and the new problems we find while we try to solve them.

Figure 1.7 shows the production matrices for $n = 6$ vertices for non-crossing partitions (b), geometric graphs (c) and connected graphs (d). In the case of triangulations (a), quadrangulations (e) and k-angulations (f), we show the production matrices for $n = 6$ internal faces.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(a) Triangulations

$$\begin{pmatrix} 2 & 4 & 8 & 16 & 32 & 64 \\ 2 & 2 & 4 & 8 & 16 & 32 \\ 0 & 2 & 2 & 4 & 8 & 16 \\ 0 & 0 & 2 & 2 & 4 & 8 \\ 0 & 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{pmatrix}$$

(c) Geometric graphs

$$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

(e) Quadrangulations

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 8 & 16 \\ 1 & 0 & 1 & 2 & 4 & 8 \\ 0 & 1 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) Non-crossing partitions

$$\begin{pmatrix} 3 & 7 & 15 & 31 & 63 & 127 \\ 1 & 3 & 7 & 15 & 31 & 63 \\ 0 & 1 & 3 & 7 & 15 & 31 \\ 0 & 0 & 1 & 3 & 7 & 15 \\ 0 & 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

(d) Connected graphs

$$\begin{pmatrix} \binom{k-2}{k-3} & \binom{k-1}{k-3} & \binom{k}{k-3} & \binom{k+1}{k-3} & \binom{k+2}{k-3} & \binom{k+3}{k-3} \\ 1 & \binom{k-2}{k-3} & \binom{k-1}{k-3} & \binom{k}{k-3} & \binom{k+1}{k-3} & \binom{k+2}{k-3} \\ 0 & 1 & \binom{k-2}{k-3} & \binom{k-1}{k-3} & \binom{k}{k-3} & \binom{k+1}{k-3} \\ 0 & 0 & 1 & \binom{k-2}{k-3} & \binom{k-1}{k-3} & \binom{k}{k-3} \\ 0 & 0 & 0 & 1 & \binom{k-2}{k-3} & \binom{k-1}{k-3} \\ 0 & 0 & 0 & 0 & 1 & \binom{k-2}{k-3} \end{pmatrix}$$

(f) k-angulations

Figure 1.7: Production matrices for six different graph classes.

The results for the entries of the vector that counts the number of graphs for the five graph classes we study are stated in Figure 1.8. For k-angulations, the formula counts the number of k-angulations with r k-gons, on n points, and root vertex degree $j-1$, for $j=1,\dots,n-1$. The other formulas enumerate the graphs on n vertices and root vertex degree $j-1$, for $j=1,\dots,n-1$.

k-angulations	$\frac{j}{r}\binom{(k-1)r-j-1}{r-j}$
Geometric graphs	$\frac{j}{n-1}2^{n-1-k}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k-j-2}{k-j}(-1)^{n-1-k}2^k$
Connected graphs	$\frac{j}{n-1}2^{n-1}\sum_{k=0}^{n-1}\binom{n-1}{k}\left(-\frac{1}{2}\right)^k\sum_{\ell=0}^{n-j-1}\binom{n-2-k+\ell}{\ell}\binom{k+n-\ell-j-2}{n-\ell-j-1}2^\ell$
Spanning trees	$\frac{j}{n}\binom{3n-j-1}{n-j}$
Non-crossing partitions	$\frac{j}{n-1}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n-k-j-2}{n-2k-j-1}2^{n-2k-j-1}$

Figure 1.8: Entries for the vectors that enumerate the number of graphs for the five graph classes.

The recursive formulas obtained for the characteristic polynomial of the production matrices of the graph classes we study are stated in Figure 1.9.

k-angulations	$k_r(\lambda)=\left(\binom{k-2}{k-3}-\lambda\right)k_{r-1}(\lambda)-\sum_{i=2}^{r}(-1)^i\binom{k+i-3}{k-3}k_{r-i}(\lambda)$
Geometric graphs	$g_n(\lambda)=(2-\lambda)g_{n-1}(\lambda)-\sum_{i=2}^{n}(-1)^i2^{2i-1}g_{n-i}(\lambda)$
Connected graphs	$c_n(\lambda)=(3-\lambda)c_{n-1}(\lambda)-\sum_{i=2}^{n}(-1)^i(2^{i+1}-1)c_{n-i}(\lambda)$
Spanning trees	$s_n(\lambda)=(2-\lambda)s_{n-1}(\lambda)-\sum_{i=2}^{n}(-1)^i(i+1)s_{n-i}(\lambda)$
Non-crossing partitions	$b_n(\lambda)=-\lambda b_{n-1}(\lambda)-\frac{1}{4}\sum_{i=2}^{n}(-2)^ib_{n-i}(\lambda)$

Figure 1.9: Recursive formulas for the characteristic polynomials for the production matrices.

Finally, the solutions for the recursive formulas of the characteristic polynomials stated in Figure 1.9 are given in Figure 1.10.

k-angulations	$k_r(\lambda)=\sum_{\ell=0}^{r}(-1)^\ell\binom{(k-2)\ell+(k-2)}{r-\ell}\lambda^\ell$
Geometric graphs	$g_n(\lambda)=\sum_{t=0}^{n}\sum_{k=0}^{n}(-1)^k\binom{k}{t}\binom{t+1}{n-k}2^{2n-t-k}\lambda^t$
Connected graphs	$c_n(\lambda)=\sum_{t=0}^{n}\left(\sum_{\ell=0}^{t}\binom{t}{\ell}(-1)^t2^{t-\ell}3^{n+2\ell-3t-2}\left[2\binom{\ell}{n+2\ell-3t-2}+9\binom{\ell+1}{n+2\ell-3t}\right]\right)\lambda^t$
Spanning trees	$s_n(\lambda)=\sum_{t=0}^{n}(-1)^t\binom{2t+2}{n-t}\lambda^t$
Non-crossing partitions	$b_n(\lambda)=\sum_{t=0}^{n}\sum_{k=0}^{n}\sum_{\ell=0}^{k}\binom{k}{t}\binom{t}{\ell}(-1)^k2^{2k-n+t-2\ell}\left[4\binom{k-t}{2k-n-\ell+1}+\binom{k-t}{2k-n-\ell}\right]\lambda^t$

Figure 1.10: Characteristic polynomials for the production matrices.

Chapter 2

Hessenberg-Toeplitz matrices

This chapter is devoted to find the characteristic polynomial and the form of an eigenvector of Hessenberg-Toeplitz matrices. This class of matrices is encountered in various scientific and engineering applications such as [4, 5, 32].

Many linear algebra algorithms require significantly less computational effort when applied to triangular matrices, and this improvement often carries over to Hessenberg matrices as well. A Hessenberg matrix is a special kind of square matrix, one that is "almost" triangular. To be exact, an upper Hessenberg matrix has zero entries below the first subdiagonal, and a lower Hessenberg matrix has zero entries above the first superdiagonal. Almost all the production matrices we obtain are upper Hessenberg matrices.

We are interested in this type of matrices because most of the production matrices of this thesis are upper Hessenberg-Toeplitz. Thus, these results allow us to find formulas for the characteristic polynomial and the eigenvectors of the production matrices for the classes of graphs we study.

Definition 3 ***(Toeplitz matrix)***

A Toeplitz matrix is an $n \times n$ matrix T_n of the form

$$T_n = \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & t_{n-1} \\ t_{-1} & t_0 & t_1 & \cdots & t_{n-2} \\ t_{-2} & t_{-1} & t_0 & \cdots & t_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{1-n} & t_{2-n} & t_{3-n} & \cdots & t_0 \end{pmatrix}$$

Definition 4 ***(Upper Hessenberg matrix)***

An Upper Hessenberg matrix is an $n \times n$ matrix H_n which has zero entries in the lower triangular part below the first subdiagonal, i.e., a matrix of the form

$$H_n = \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} & \cdots & h_{1,n} \\ h_{2,1} & h_{2,2} & t_{2,3} & \cdots & t_{2,n} \\ 0 & h_{3,2} & h_{3,3} & \cdots & t_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{n,n} \end{pmatrix}$$

In this thesis, we work with the class of Upper Hessenberg-Toeplitz matrices, i.e., matrices of the form

$$A_n = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & a_{-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix}$$

The determinant of Hessenberg-Toeplitz matrices has already been determined [11]. However, we obtain a formula for the characteristic polynomial of A_n, which is the determinant of $A_n - \lambda I_n$, being λ an eigenvalue of A_n and I_n the $n \times n$ identity matrix.

Theorem 6 *The characteristic polynomial $d_n(\lambda)$ of an Upper Hessenberg-Toeplitz matrix A_n satisfies the recurrence relation*

$$d_n(\lambda) = (a_0 - \lambda)d_{n-1}(\lambda) + \sum_{i=2}^{n} (-1)^{i+1} a_{i-1} a_{-1}^{i-1} d_{n-i}(\lambda)$$

Proof: To calculate $d_n(\lambda)$, we develop the determinant of $(A_n - \lambda I_n)$ with respect to the last row, and obtain the recurrence relation

$$d_n(\lambda) = det(A_n - \lambda I_n) = (a_0 - \lambda)d_{n-1}(\lambda) - a_{-1} det(L_{n-1,1}) \tag{2.1}$$

where $L_{n-\ell,i}$ is the $(n-\ell) \times (n-\ell)$ matrix

$$L_{n-\ell,i} = \begin{pmatrix} a_0 - \lambda & a_1 & a_2 & a_3 & \cdots & a_{n-2-\ell} & a_{n-1} \\ a_{-1} & a_0 - \lambda & a_1 & a_2 & \cdots & a_{n-3-k} & a_{n-2} \\ 0 & a_{-1} & a_0 - \lambda & a_1 & \cdots & a_{n-4-k} & a_{n-3} \\ 0 & 0 & a_{-1} & a_0 - \lambda & \cdots & a_{n-5-k} & a_{n-4} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & a_0 - \lambda & a_{i+1} \\ 0 & 0 & 0 & 0 & \cdots & a_{-1} & a_i \end{pmatrix}$$

In the same way as for the determinant of $(A_n - \lambda I_n)$, we obtain the recurrence relation for the determinant of the matrix $L_{n-1,1}$

$$det(L_{n-1,1}) = a_1 d_{n-2}(\lambda) - a_{-1} det(L_{n-2,2}) \tag{2.2}$$

From Equations (2.1) and (2.2) we obtain

$$d_n(\lambda) = (a_0 - \lambda)d_{n-1}(\lambda) - \sum_{i=2}^{n} (-1)^i a_{-1}^{i-1} a_{i-1} d_{n-i}(\lambda)$$

□

The second result of this chapter, is a formula for the entries of an eigenvector associated to an eigenvalue λ of an Upper Hessenberg-Toeplitz matrix, and is given by the following theorem.

Theorem 7 *Let $x = (x_{n-1}, x_{n-2}, \dots, x_0)$ be an eigenvector of an Upper Hessenberg-Toeplitz matrix A_n associated to an eigenvalue λ. Then the entries of the vector x are of the form:*

$$x_i = \left(\frac{-1}{a_{-1}}\right)^i d_i(\lambda) x_0$$

$\forall i = 1, \dots, n-1$, *where $d_i(\lambda)$ is the characteristic polynomial of A_i.*

Proof: We have to solve the system of equations $A_n x = \lambda x$, where λ is the eigenvalue associated to the eigenvector x.

We will prove the theorem by induction on the i-th entry of the vector.

Base case: the cases $i = 1$ and $i = 2$ are easily verified:
If we multiply the last row of the matrix A_n by the eigenvector x, we obtain that

$$a_{-1}x_1 + a_0 x_0 = \lambda x_0 \implies x_1 = \frac{(\lambda - a_0)}{a_{-1}} x_0 = \frac{-d_1(\lambda)}{a_{-1}} x_0$$

The case $i = 2$ is analogous to the previous one:

$$a_{-1}x_2 + a_0 x_1 + a_1 x_0 = \lambda x_1 \implies x_2 = \frac{(\lambda - a_0)x_1 - a_1 x_0}{a_{-1}} = \left[\frac{\frac{(a_0-\lambda)d_1(\lambda)}{a_{-1}} - a_1}{a_{-1}}\right] x_0 = \frac{d_2(\lambda)}{a_{-1}^2} x_0$$

Inductive hypothesis: we will suppose that $x_i = (-1)^i \frac{d_i(\lambda)}{a_{-1}^i} x_0$ for all $i < n-2$.

Inductive step: if we multiply the second row of the production matrix A_n by the eigenvector x, we have that:

$$a_{-1}x_{n-1} + a_0 x_{n-2} + a_1 x_{n-3} + a_2 x_{n-4} + \cdots + a_{n-2}x_0 = \lambda x_{n-2} \implies$$
$$\implies x_{n-1} = \frac{(\lambda - a_0)x_{n-2} - a_1 x_{n-3} - a_2 x_{n-4} - \cdots - a_{n-2}x_0}{a_{-1}}$$

Then, by the induction hypothesis, this is equal to

$$a_{-1}x_{n-1} = \left[\frac{(-1)^{n-2}}{a_{-1}^{n-2}}(\lambda - a_0)d_{n-2}(\lambda) - a_1\frac{(-1)^{n-3}}{a_{-1}^{n-3}}d_{n-3}(\lambda) - \cdots - a_{n-2}\right] x_0 =$$
$$= \frac{(-1)^{n-1}}{a_{-1}^{n-2}}\left[(a_0 - \lambda)d_{n-2}(\lambda) + \sum_{i=2}^{n-1}(-1)^{i+1}a_{i-1}a_{-1}^{i-1}d_{n-i-1}(\lambda)\right] x_0 \implies$$
$$\implies x_{n-1} = \frac{(-1)^{n-1}}{a_{-1}^{n-1}}\left[(a_0 - \lambda)d_{n-2}(\lambda) + \sum_{i=2}^{n-1}(-1)^{i+1}a_{i-1}a_{-1}^{i-1}d_{n-i-1}(\lambda)\right] x_0$$

And by Theorem 6, this is equal to $\frac{(-1)^{n-1}}{a_{-1}^{n-1}} d_{n-1}(\lambda) x_0$. So the last part of the proof consists in developing the first row of the system of equations given by $a_0 x_{n-1} + a_1 x_{n-2} + a_2 x_{n-3} + a_3 x_{n-4} + \cdots + a_{n-1} x_0 = \lambda x_{n-1}$.

$$\begin{aligned} a_0 x_{n-1} + a_1 x_{n-2} + a_2 x_{n-3} + a_3 x_{n-4} + \cdots + a_{n-1} x_0 = \lambda x_{n-1} &\implies \\ \implies (a_0 - \lambda) x_{n-1} + a_1 x_{n-2} + a_2 x_{n-3} + a_3 x_{n-4} + \cdots + a_{n-1} x_0 = 0 \end{aligned}$$

Now, if we substitute each x_i by its value, we get that this is equal to

$$\left[(a_0 - \lambda) \frac{(-1)^{n-1}}{a_{-1}^{n-1}} d_{n-1}(\lambda) + a_1 \frac{(-1)^{n-2}}{a_{-1}^{n-2}} d_{n-2}(\lambda) + a_2 \frac{(-1)^{n-3}}{a_{-1}^{n-3}} d_{n-3}(\lambda) + \cdots + a_{n-1} \right] x_0 = 0$$

But this is true because

$$\begin{aligned} (a_0 - \lambda) \frac{(-1)^{n-1}}{a_{-1}^{n-1}} d_{n-1}(\lambda) + a_1 \frac{(-1)^{n-2}}{a_{-1}^{n-2}} d_{n-2}(\lambda) + a_2 \frac{(-1)^{n-3}}{a_{-1}^{n-3}} d_{n-3}(\lambda) + \cdots + a_{n-1} = \\ = \frac{(-1)^{n-1}}{a_{-1}^{n-1}} \left[(a_0 - \lambda) d_{n-1}(\lambda) + \sum_{i=2}^{n} (-1)^{i+1} a_{i-1} a_{-1}^{i-1} d_{n-i}(\lambda) \right] = \frac{(-1)^{n-1}}{a_{-1}^{n-1}} d_n(\lambda) \end{aligned}$$

And this is equal to 0 if $d_n(\lambda) = 0$, but this is also true due to the fact that $d_n(\lambda)$ is the characteristic polynomial of A_n. $\square$

Chapter 3

k-angulations

The third chapter of the thesis is dedicated to the class of k-angulations [47, 49]. A k-angulation is a 2-connected graph where all interior faces are k-gons.

For 3-gons we fall into the well-known case of triangulations. It is known that the number of triangulations of a set of $n+2$ points in convex position is counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Catalan numbers are a sequence of positive integers that appear in many counting problems in combinatorics [53, 54]. They count certain types of lattice paths, permutations, binary trees, and many other combinatorial objects. They satisfy the following fundamental recurrence relation.

$$C_n = \frac{2(2n-1)}{n+1} C_{n-1}$$

The production matrix for triangulations T_n is given for instance in [14], also the eigenvalues of the matrix T_n have been determined by Chow [6], and a formula for the characteristic polynomial of T_n using Chebyshev polynomial has been developed, see [17], also in [29] we can find another formula for the characteristic polynomial. So, having an expression for the characteristic polynomial allows us to apply Cayley-Hamilton's theorem [29] to get combinatorial identities.

The equivalence of several counting problems has been observed many times, by showing a bijection between these types of problems. One of these results is the one that establishes a bijection between the triangulations of a set of n points in convex positions and rooted binary trees, where a rooted binary tree is a tree with one root node, and where each node has either zero or two branches descending from it. As it was proved in many occasions [8, 12, 51], rooted binary trees with n internal nodes are also counted with Catalan numbers; a node is internal if it has two nodes descending from it.

One of the bijections between triangulations of a set of n points in convex positions and rooted binary trees on $n+1$ leaves (given in [8]) is featured in Figure 3.1. Under this bijection, each leaf of the rooted binary tree corresponds to a boundary edge in the triangulation and internal nodes of the binary tree correspond to internal faces of the triangulation. We consider the edge of the polygon corresponding to the root of the tree to be marked giving rise to a marked triangulation.

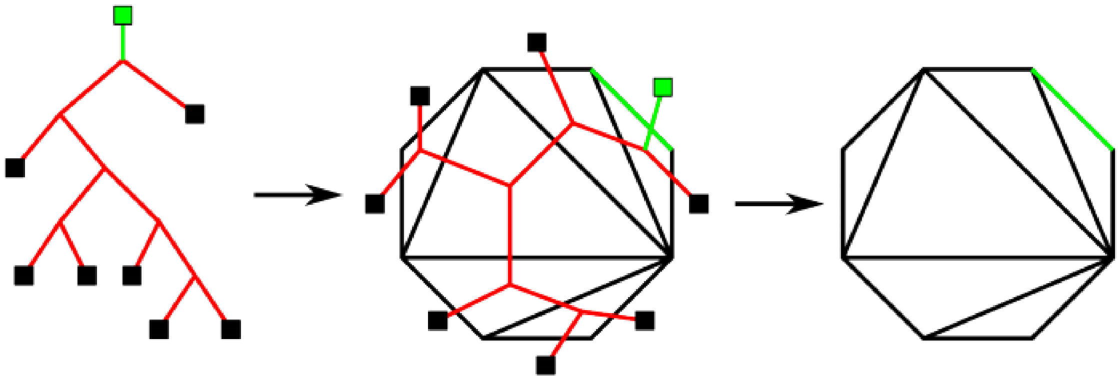

Figure 3.1: Converting a rooted binary tree with 7 leaves into its corresponding triangulation of an octagon with a marked edge.

We will count triangulations as a function of the number of non-rooted boundary edges (or by leaves in the corresponding rooted binary tree). Thus, a triangulation of size n means a triangulation of an $(n+1)$-gon or with n leaves in the underlying binary tree.

Also, bijections between k-angulations and $(k-1)$-ary trees and between $(k-1)$-ary trees and spanning trees are well-known. Thus, it will not be surprising to see in the following sections that the production matrix for quadrangulations coincides with the known production matrix for spanning trees. Hence, our work has consisted in finding analogous results for k-angulations for a given $k \geq 4$, but working with a new definition of the degree for the production matrix.

The first section is devoted to determine the main results for quadrangulations, i.e. $k = 4$.

3.1 Quadrangulations

3.1.1 Production matrix

We will introduce this framework about production matrices and generating trees with our first graph class, the quadrangulations. We must say that in this section, we count the number of quadrangulations on r 4-gons, i.e on $n = 2r + 2$ vertices. Our production matrix is devised by using a surjective mapping of the graphs on $(r + 1)$ 4-gons to the graphs on r 4-gons. Let v^r be the vector that counts the number of quadrangulations on r 4-gons and $2r+2$ vertices, and let σ be a quadrangulation with point set $(q_1, \ldots, q_{2r+2})$. Then the entries of the vector v^r will depend on properties of the root vertex q_{2r+2} in this mapping.

This mapping is based on a graph operation called flipping. The flip operation is defined in [39] as follows: suppose that a quadrangulation σ has an hexagonal region $q_1q_2q_3q_4q_5q_6$ with unique diagonal q_1q_4 and no inner vertices. The diagonal flip is an operation replacing the diagonal q_1q_4 with q_2q_5, or with q_3q_6 (see Figure 3.2). If a diagonal flip yields multiple edges or loops, then we do not apply it. This operation clearly transforms a quadrangulation to another one.

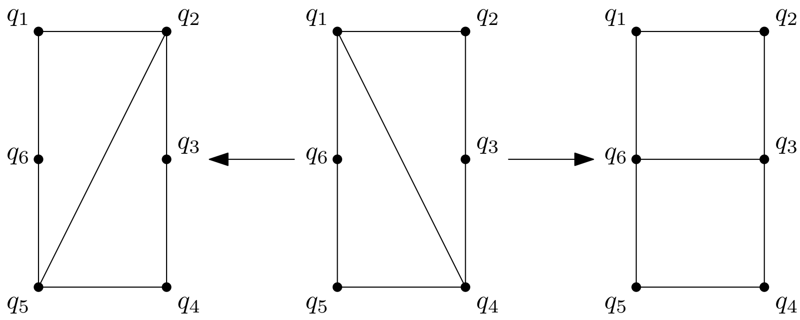

Figure 3.2: The diagonal flip.

Once we have defined the tool needed to define the mapping, let us apply it to quadrangulations. Let σ be a quadrangulation with a set of points in convex position $\{q_1, ..., q_{2r+4}\}$ numbered counter-clockwise, where the vertex q_{2r+4} is the root vertex and its degree is defined as the number of incident edges to it, minus 2.

We define the tree of quadrangulations by using the following mapping between the set $\mathfrak{Q}_{r+1}$ of quadrangulations with $(r+1)$ 4-gons and $\mathfrak{Q}_r$. Let $d(q_i)$ be the degree of the vertex q_i, in a quadrangulation $\sigma' \in \mathfrak{Q}_{r+1}$ ignoring the boundary edges. We can obtain a quadrangulation $\sigma \in \mathfrak{Q}_r$ from σ' following this procedure: if $d(q_{2r+4}) + d(q_{2r+3}) = 0$, we just delete both vertices, otherwise we flip all the edges incident to q_{2r+3} to q_{2r+2}, then we flip the edges incident to q_{2r+4} to q_{2r+2}, in order to have no incident edges to q_{2r+4} and q_{2r+3}; then we can delete both vertices (see Figure 3.3). We call σ' the parent of σ and σ the child of σ'.

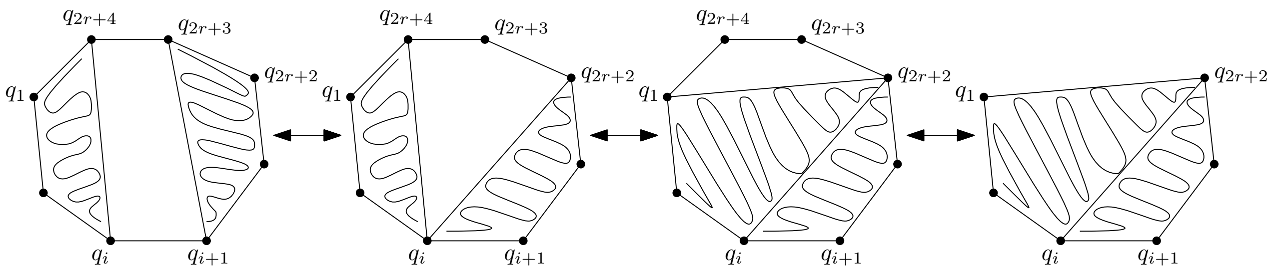

Figure 3.3: Construction of a child σ (right) of σ' (left).

Going the other direction, observe that the number of children of σ' depends on the vertex degree $d(q_{2r+2})$ of q_{2r+2}, and that a child can always be obtained by adding q_{2r+3} and q_{2r+4} and flipping a subset of edges that are incident to q_{2r+2} (see Figure 3.3) such that they are incident to q_{2r+3} and q_{2r+4}.

Observe that each quadrangulation is generated exactly once. Once we know how to derive $\mathfrak{Q}_{r+1}$ from $\mathfrak{Q}_r$, we are able to produce the generating tree of quadrangulations given by Figure 3.4.

Let v^i_j be the number of quadrangulations with i 4-gons such that $d(q_{2i+2}) = j - 1$, and denote with v^i the vector whose j-th entry is v^i_j. Thus, to derive v^{i+1} from v^i, we first observe that we can add the face with vertices q_1, q_{2r+2}, q_{2r+3} and q_{2r+4} to any quadrangulation of $\mathfrak{Q}_r$, and flip the edges that are incident to the former root vertex q_{2r+2}; then, they are incident to q_{2r+3} or q_{2r+4}.

Thus, considering v^i as an r-dimensional vector with $r > i$, we show that v^{i+1} can be obtained by multiplying v^i by a production matrix Q_r, that is, $v^{i+1} = Q_r v^i$.

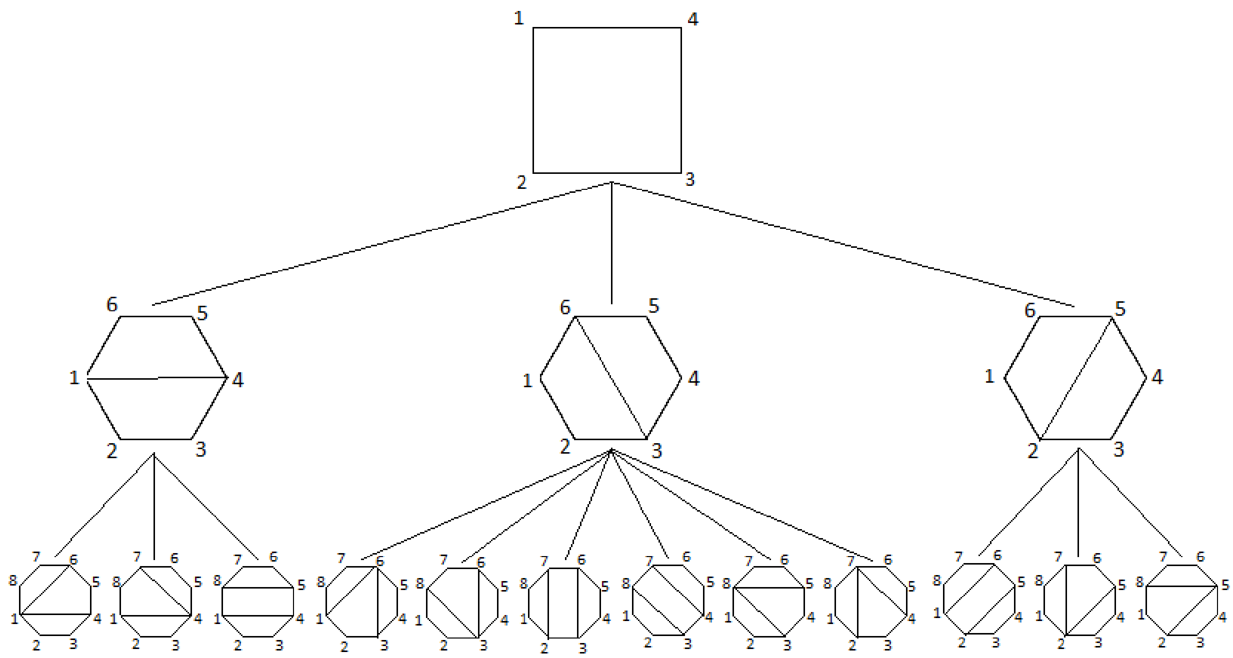

Figure 3.4: First levels of the generating tree of quadrangulations.

Theorem 8 *The following $r \times r$ matrix Q_r is a production matrix for quadrangulations of point sets in convex position.*

$$Q_r = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & \cdots & r+1 \\ 1 & 2 & 3 & 4 & 5 & \cdots & r \\ 0 & 1 & 2 & 3 & 4 & \cdots & r-1 \\ 0 & 0 & 1 & 2 & 3 & \cdots & r-2 \\ 0 & 0 & 0 & 1 & 2 & \cdots & r-3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

Proof: Following the notion of the flip operation defined above, we want to know the number of vertices with root vertex of degree $j-1$ on $r+1$ 4-gons, from the knowledge about v^r. Assume we are given a quadrangulation $\sigma \in \mathfrak{Q}_r$, and the vertex q_{2r+2} has degree $d(q_{2r+2}) = t$. Now, we produce $\sigma' \in \mathfrak{Q}_{r+1}$ by adding two new vertices q_{2r+3} and q_{2r+4} to the quadrangulation creating a new face with vertices q_1, q_{2r+2}, q_{2r+3} and q_{2r+4}. If we want $d(q_{2r+4}) = 0$, we have the possibility of flippling a subset of the t edges incident to q_{2r+2} in σ' except the ones joining it with q_{2r+1} and q_{2r+3}, i.e. replacing an edge $q_{2r+2}q_i$ by the edge $q_{2r+3}q_{i+1}$ for all q_i adjacent to q_{2r+2} in σ and obtaining t quadrangulations (see Figure 3.4). In general, if we want $d(q_{2r+4}) = m$, $m \leq t-1$, we have to flip m edges from q_{2r+2} to q_{2r+4}, and the remaining of the $t-m$ edges of q_{2r+2}, we can leave them incident to q_{2r+2} or flip a subset of them from q_{2r+2} to q_{2r+3}. Note that we obtain one quadrangulation less than if we want degree $m-1$ on q_{2r+4}. This leads us to the production matrix for quadrangulations Q_r. $\square$

The production matrix Q_r can also be obtained by the following rule: if the root vertex q_{2r+2} of a quadrangulation has degree k, then it has $k+2$ successors (children) of degree 0, $k+1$ of degree 1, etc., and one of degree $k+1$. Hence this succession rule is equivalent to the generating tree of quadrangulations and the production matrix Q_r is the corresponding AGT matrix (see Definition 2 in Chapter 1).

The production matrix Q_r also corresponds to a Riordan Array, see [38]. Applying the results in [38] one obtains that the number of quadrangulations on r 4-gons, with root vertex of degree $j-1$ is equal to $\frac{j}{r}\binom{3r-j-1}{r-j}$, given by Sequence $A071948$ in the On-Line Encyclopedia of Integer Sequences [41]. A proof is also given in Theorem 16, Section 3.3.1, for k-angulations using $k=4$.

Let v^i be the vector of quadrangulations (that is, v^i is the first column of a power of Q_r for $i<r$; the sum of the elements of v^i is the number of quadrangulations with i 4-gons). The first vectors are:

$$v^1 = \begin{pmatrix} 1\\0\\0\\0\\0\\ \vdots \end{pmatrix} \qquad v^2 = \begin{pmatrix} 2\\1\\0\\0\\0\\ \vdots \end{pmatrix} \qquad v^3 = \begin{pmatrix} 7\\4\\1\\0\\0\\ \vdots \end{pmatrix} \qquad v^4 = \begin{pmatrix} 30\\18\\6\\1\\0\\ \vdots \end{pmatrix} \tag{3.1}$$

The number of quadrangulations given by the sum of the vector entries in (3.1) was also calculated by Noy and Flajolet [18] and by Noy in [40] for spanning trees. These numbers are $\{1,3,12,55,273,1428,\ldots\}$, see Sequence $A001764$ in The On-Line Encyclopedia of Integer Sequences [41]. They are given by the equation

$$\frac{1}{3n+1}\binom{3n+1}{n}$$

The numbers of the form $\frac{r}{pn+1}\binom{pn+1}{n}$ are called Pfaff-Fuss-Catalan sequence $A_n(p,r)$. In our case, we have that $p=3$ and $r=1$ [20]. The Pfaff-Fuss-Catalan numbers when $p=3$ consider exactly the same recurrence as in the Catalan triangle,

$$B(n,k) = B(n-1,k) + B(n,k-1)$$

letting the array grow, not in 2, but in 3 dimensions. More precisely, the sequence $B_3(n,k,\ell)$, introduced in [1], is indexed by a positive integer n and non-negative integers k and ℓ, and defined recursively by:

$$B_3(n,k,\ell) = B_3(n-1,k,\ell) + B_3(n,k-1,\ell) + B_3(n,k,\ell-1) - B_3(n,k-1,\ell-1)$$

As the numbers that counts quadrangulations and the numbers counting spanning trees are the same, we present a bijection between these two graph classes.

3.1.2 A bijection between quadrangulations and spanning trees

The bijection between binary trees and triangulations shown in the introduction of Chapter 3 can be adapted to a bijection between quadrangulations and ternary trees in a straight-forward way. In addition, there is the strong connection between the number of non-crossing trees N_{n+1} and the number of ternary trees T_n, $N_{n+1} = T_n$ (see [42]). In fact, a bijection between these families of trees was given in [42]. This classical correspondence proves that the number P_{n+1} of planted plane trees with $n+1$ nodes equals the number B_n of binary trees with n internal vertices, given by Catalan numbers.

Formally, the bijection ϕ from ternary trees to non-crossing trees proposed in [42] can be described in the following way. Let T be a ternary tree, then they construct a non-crossing tree $N = \phi(T)$:

1. The vertices of T are the vertices of N with the root deleted.
2. The root of T is the first right child of the root of N.
3. Vertex v is a left child of vertex w in T if and only if v is the first left child of w in N.
4. Vertex v is a middle child of vertex w in T if and only if v is the first right child of w in N.
5. Vertex v is a right child of vertex w in T if and only if v is the brother to the right of w in N.

In [42] it is proved that ϕ indeed defines a bijection. Thus, the number of quadrangulations with r 4-gons is the same as the number of straight line crossing-free spanning trees on $r+1$ points in the plane.

3.1.3 Characteristic polynomial

Knowing the characteristic polynomial of a production matrix is interesting to get combinatorial identities via Cayley-Hamilton's theorem. For instance, the solutions of the characteristic polynomial $q_r(\lambda)$ of Q_r give the eigenvalues of the production matrix Q_r, a result yet unknown.

In the following, we present the characteristic polynomial of the production matrix Q_r. Since these polynomials were known for spanning trees [29] and the production matrix Q_r for quadrangulations and for spanning trees is the same, we omit the proof here, and it is proved in Theorem 17 in Section 3.3.2 for each value of k.

Let $q_r(\lambda)$ be the characteristic polynomial of Q_r. The sequence $\{q_r(\lambda)\}$ starts with

$$\begin{array}{c|c}
r=1 & -\lambda+2 \\
r=2 & \lambda^2-4\lambda+1 \\
r=3 & -\lambda^3+6\lambda^2-6\lambda \\
r=4 & \lambda^4-8\lambda^3+15\lambda^2-4\lambda \\
r=5 & -\lambda^5+10\lambda^4-28\lambda^3+20\lambda^2-\lambda \\
r=6 & \lambda^6-12\lambda^5+45\lambda^4-56\lambda^3+15\lambda^2
\end{array}$$

Theorem 9 ***(Huemer et al. [28])***
The characteristic polynomial $q_r(\lambda)$ of the matrix Q_r satisfies the recurrence relation

$$q_r(\lambda) = (2-\lambda)q_{r-1}(\lambda) + \sum_{i=2}^{r}(-1)^{i+1}(i+1)q_{r-i}(\lambda)$$

and the solution of this recurrence relation with initial condition $q_0(\lambda) = 1$ is

$$q_r(\lambda) = \sum_{k=0}^{r}\binom{2k+2}{r-k}(-1)^k\lambda^k$$

Corollary 1 ***(Huemer et al. [28])***
Let $r \geq 3$

$$q_r(\lambda) = -\lambda(q_{r-1}(\lambda) + 2q_{r-2}(\lambda) + q_{r-3}(\lambda))$$

where q_i is the characteristic polynomial of the production matrix Q_i

Having an expression for the characteristic polynomial allows us to apply Cayley-Hamilton's theorem (see Theorem 4), which in this case tells us that $\sum_{j=0}^{r}\binom{2j+2}{r-j}(-1)^j(Q_r)^j = \mathbf{0}_r$, where $\mathbf{0}_r$ denotes the $r \times r$ zero matrix. This gives a relation among the numbers of quadrangulations with root vertex of degree k. For example, for $k = 0$ (entry $(1,1)$ of Q_r) we obtain the following relation:

Proposition 1 *Let $r \geq 2$, then:*

$$\sum_{j=0}^{r}\frac{1}{3j+1}\binom{3j+1}{j}\binom{2j+2}{r-j}(-1)^j = 0$$

3.1.4 Eigenvectors and eigenvalues of Q_r

One interesting property about counting graphs is the asymptotic growth of the number of the graphs. In the case of spanning trees (or quadrangulations), this asymptotic growth is approximated by Flajolet and Noy in [18], giving the estimation of $\left(\frac{27}{4}\right)^n$. Also, by Perron-Frobenius' theorem, we know that the largest eigenvalue of a matrix determines the growth constant, as the matrix size tends towards infinity. We do not know much more concerning the eigenvalues of Q_r. Here we claim an asymptotic result on the sum of powers of the eigenvalues of Q_r, in Theorem 10.

But first, we obtain the following result concerning the entries of an eigenvector associated to the production matrix Q_r.

Corollary 2 *Let $x = (x_{r-1}, x_{r-2}, \ldots, x_0)$ be an eigenvector of the production matrix Q_r associated to an eigenvalue λ. Then the entries of the vector x are of the form:*

$$x_i = (-1)^i q_i(\lambda) x_0$$

where $q_i(\lambda)$ is the characteristic polynomial of the production matrix Q_i.

The proof of the Corollary is given by Theorem 6, taking $a_{-1} = 1$.

Now, we proceed to give a result for the eigenvalues of the production matrix Q_r of quadrangulations. Actually, we prove Equation (3.3) given below. In order for Theorem 10 to be true, we still need to verify the *canonical distribution assumption*, given by Equation (3.2). That is, it remains to verify the assumption (3.2), which is known to hold for many Toeplitz matrices.

Theorem 10 *Let $\lambda_{r,k}$ be the k-th eigenvalue of the production matrix Q_r, for $k = 0, \ldots, r-1$. Then for and integer $s \geq 0$*

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r-1} \lambda_{r,k}^s = \binom{3s-1}{s}$$

if the canonical distribution assumption holds.

Proof: We follow the exposition of [21]. The production matrix Q_r is a Toeplitz matrix whose first row has elements $t_0 = 2, t_1 = 3, t_2 = 4$ etc. The second row has elements $t_{-1} = 1, t_0 = 2, t_1 = 3$ etc. Hence, we have $t_j = j + 2$ if $j = -1, \ldots, r-1$, and $t_j = 0$ otherwise. Consider the Fourier series

$$f_r(\lambda) = \sum_{k=-\infty}^{\infty} t_k e^{ik\lambda} = \sum_{k=-1}^{r-1} (k+2) e^{ik\lambda}$$

It follows from [21] that

$$t_k = \frac{1}{2\pi} \int_0^{2\pi} f_r(\lambda) e^{-ik\lambda} \, d\lambda$$

Thus, the sequence $\{t_k\}$ determines the function $f_r(\lambda)$ and vice versa.
To prove the result we need the assumption

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=0}^{r-1} \lambda_{r,k}^s = \frac{1}{2\pi} \int_0^{2\pi} f_r(\lambda)^s \, d\lambda \tag{3.2}$$

Then we need to prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=-1}^{r-1} (k+2) e^{ik\lambda} \right)^s \, d\lambda = \binom{3s-1}{s} \tag{3.3}$$

It is known that if and only if $k = 0$, for $k \in \{-1, 0, 1, 2, \ldots\}$, then

$$\frac{1}{2\pi} \int_0^{2\pi} (k+2) e^{ik\lambda} \, d\lambda \neq 0$$

From the multinomial theorem, we know that

$$(x_1 + \ldots + x_m)^n = \sum_{k_1+\ldots+k_m=n} \binom{n}{k_1, \ldots, k_m} \prod_{t=1}^{m} x_t^{k_t}$$

In our case, we have that the multinomial is equal to the following equation:

$$\sum_{a_1+\ldots+a_{r+1}=s} \binom{s}{a_1, \ldots, a_{r+1}} \prod_{t=-1}^{r-1} ((t+2)e^{it\lambda})^{a_t}$$

So, the integral of Equation (3.3) is different from 0 when $a_1 + a_2 + \ldots + a_s = 0$, with $a_i \in \{-1, 0, 1, \ldots s\}$. Since $\sum_{i=1}^{s} a_i = 0$, then $\sum_{i=1}^{s}(a_i + 2) = 2s$. Hence, we look for all compositions of $2s$ into s blocks. The integral is then $\sum_{a_1,\ldots,a_{r+1}=s}^{s} \prod_{i=1}^{s}(a_i + 2)$. Thus, according to [25, (v)], the sum of the products of the parts in the k-part compositions of n is the binomial coefficient $\binom{n+k-1}{n-k}$. If we take $n = 2s$ and $k = s$ we obtain the desired formula.

For instance, if $s = 3$, we have the following compositions: $(0, 0, 0), (-1, 0, 1)$ (and its six permutations) and $(-1, -1, 2)$ (and its three permutations), which correspond (summing up 2 to each term) to $(2, 2, 2), (1, 2, 3)$ (and its six permutations) and $(1, 1, 4)$ (and its three permutations). We can see that we have compositions of $2s = 6$ into $s = 3$ blocks. And for each block we need the product of their terms, which is $2 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 3 \cdot 6 + 1 \cdot 1 \cdot 4 \cdot 3 = 56$, the solution of the integral for $s = 3$.

□

For the first values of s we get (taking r sufficiently large) the sequence $\{2, 10, 56, 330, 2002, 12376, 77520, \ldots\}$, which is Sequence $A165817$ in the On-Line Encyclopedia of Integer Sequences [41].

3.1.5 Other properties

As we are interested in finding relations among the vector that counts quadrangulations, the characteristic polynomial $q_r(\lambda)$, and the eigenvectors of the production matrix Q_r, we begin searching for some new properties.

In matrix computations there are matrices that play a fundamental role. One class of these matrices are called *Krylov matrices*. They are key tools in understanding and developing numerical methods for solving eigenvalue problems and systems of linear equations, including the QR algorithm and Krylov subspace methods, see e.g., [19, 22, 62]. Let $A \in \mathbb{C}^{n,n}$. Then the Krylov matrix X of the matrix A generated by a vector $b \in \mathbb{C}^n$ is given by

$$X = [b \quad Ab \quad \cdots \quad A^{n-1}b] \in \mathbb{C}^{n,n}$$

In our case, the matrix A would be the production matrix Q_r of quadrangulations and we can take the vector b as the vector $v^1 = (1, 0, 0, 0, \ldots, 0)^\top$. So, we obtain the following Krylov matrix:

$$X_r = \begin{pmatrix} 1 & 2 & 7 & 30 & 143 & \cdots & v_1^r \\ 0 & 1 & 4 & 18 & 88 & \cdots & v_2^r \\ 0 & 0 & 1 & 6 & 33 & \cdots & v_3^r \\ 0 & 0 & 0 & 1 & 8 & \cdots & v_4^r \\ 0 & 0 & 0 & 0 & 1 & \cdots & v_5^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where each entry v_j^i is the number of quadrangulations on i 4-gons and root vertex of degree $j-1$.

Now, suppose that the characteristic polynomial of a matrix A is $r(\lambda) = det(A-\lambda I) = \lambda^n - c_n\lambda^{n-1} - \cdots - c_2\lambda - c_1$. Then, we define the companion matrix C of A as:

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & c_1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & c_2 \\ 0 & 1 & 0 & 0 & \cdots & 0 & c_3 \\ 0 & 0 & 1 & 0 & \cdots & 0 & c_4 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & c_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 1 & c_n \end{pmatrix}$$

One of the properties of this companion matrix is that the eigenvalues associated to it are the solutions of the characteristic polynomial $r(\lambda)$. So, determining the eigenvalues of C is equivalent to determining the roots of $r(\lambda)$, i.e., the eigenvalues of the matrix A.

Let us look at the companion matrix of Q_r. Since the characteristic polynomial of Q_r is given by Theorem 9 as

$$q_r(\lambda) = \sum_{k=0}^{r} \binom{2k+2}{r-k} (-1)^k \lambda^k$$

we obtain the following companion matrix for Q_r:

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & -\binom{2}{r} \\ 1 & 0 & 0 & 0 & \cdots & 0 & \binom{4}{r-1} \\ 0 & 1 & 0 & 0 & \cdots & 0 & -\binom{6}{r-2} \\ 0 & 0 & 1 & 0 & \cdots & 0 & \binom{8}{r-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & (-1)^{r-1}\binom{2(r-1)}{2} \\ 0 & 0 & 0 & 0 & \cdots & 1 & (-1)^r\binom{2r}{1} \end{pmatrix}$$

Proposition 2 *Let A, B and T be three matrices such that $AT = TB$ and T is invertible, and let x be an eigenvector of A, then $z = T^{-1}x$ is an eigenvector of B.*

Proof:

$$AT = TB \Longrightarrow A = TBT^{-1}$$

As x is an eigenvector of A, we have that $Ax = \lambda x$, and then,

$$TBT^{-1}x = \lambda x \Longrightarrow BT^{-1}x = T^{-1}\lambda x \Longrightarrow B(T^{-1}x) = \lambda(T^{-1}x)$$

Then $T^{-1}x$ is an eigenvector of B. □

Using Proposition 2 and the relation $Q_r X_r = X_r C$ [63], where X_r is the Krylov matrix of Q_r and C the companion matrix of Q_r, we can calculate the eigenvectors of C by calculating the inverse of the matrix X_r and multiplying it by an eigenvector of the production matrix Q_r given by Theorem 2.

Theorem 11 *Let X_r be the Krylov matrix associated to the production matrix Q_r, then the inverse of X_r has the following form:*

$$X_r^{-1} = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & \cdots & x_{1,n} \\ 0 & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & \cdots & x_{2,n} \\ 0 & 0 & x_{3,3} & x_{3,4} & x_{3,5} & \cdots & x_{3,n} \\ 0 & 0 & 0 & x_{4,4} & x_{4,5} & \cdots & x_{4,n} \\ 0 & 0 & 0 & 0 & x_{5,5} & \cdots & x_{5,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & x_{n,n} \end{pmatrix}$$

with entries $x_{i,j} = (-1)^{i-j}\binom{2i}{j-i}$

Proof: The proof is based on Riordan Arrays, following results in [36] and [52]. According to [36], every Riordan Array induces the two-parameters basic identity:

$$\sum_{j=k}^{n} d_{n,j} d^*_{j,k} = \delta_{n,k}$$

where $d_{i,j}$ is the generic element of a Riordan Array, the superscripted asterisk denotes the following quantity related to the inverse Riordan Arrays

$$d^*_{r,k} = \frac{1}{r}[t^{r-k}]\left(\frac{k}{d(t)} - \frac{td'(t)}{d(t)^2}\right)\left(\frac{1}{h(t)}\right)^r$$

and $\delta_{n,k}$ is the Kronecker delta.

In our case, we need to prove that $X_r \cdot X_r^{-1} = I_r$, where I_r is the $r \times r$ identity matrix, by proving that $\sum_{h=k}^{r} v_k^h(-1)^{h-k}\binom{2h}{r-h} = \delta_{r,k}$. Recall that v^h is the vector that counts the number of quadrangulations on h points, and its entries are given by $d_{r,k}$.

Theorem 2.1 in [52] gives a formula for calculating the elements of a given Riordan Array. If we consider the binomial coefficient $\binom{2k}{r-k}$ we have the infinite array given by the Riordan Array $D = (d(t), h(t)) = (1, (1+t)^2)$. Hence, we have to prove that $d^*_{r,k} = v_k^r(-1)^{r-k}$. We can prove this result with the formula given by [36, Theorem 2.2]:

$$d^*_{r,k} = \frac{1}{r}[t^{r-k}]\left(\frac{k}{d(t)} - \frac{td'(t)}{d(t)^2}\right)\left(\frac{1}{h(t)}\right)^r = \frac{1}{r}[t^{r-k}]k\left(\frac{1}{1+t}\right)^{2r} =$$
$$= \frac{k}{r}[t^{r-k}]\sum_{\ell=0}^{\infty}\binom{2r+\ell-1}{\ell}(-t)^\ell = \frac{k}{r}\binom{3r-k-1}{r-k}(-1)^{r-k} = v^r_k(-1)^{r-k}$$

where we use the well-known identity

$$\left(\frac{1}{1-x}\right)^r = \sum_{k=0}^{\infty}\binom{r+k-1}{k}x^k \tag{3.4}$$

□

As we can observe, each row i of the inverse of the Krylov matrix X_r corresponds to the $(2i+1)$-th row of the Pascal triangle (see for instance Sequence $A034870$ in the On-Line Encyclopedia of Integer Sequences [41]) with some entries multiplied by -1.

We thus have proved the following combinatorial indentity:

Corollary 3 *Let $k, r \in \mathbb{N}$, then*

$$\sum_{h=k}^{r}\frac{k}{h}\binom{3h-k-1}{h-k}\binom{2h}{r-h}(-1)^{h-k} = \delta_{r,k}$$

where $\delta_{r,k}$ is the Kronecker delta.

Let $C^\top$ be the transposed matrix of the companion matrix C, and $Q_r^\top$ the transposed matrix of Q_r. Then it is easy to verify that an eigenvector of $C^\top$ has the form $(1, \lambda, \lambda^2, \ldots, \lambda^{r-1})^\top$. It is also known that an eigenvector of $Q_r^\top$ is given by the corresponding eigenvector of Q_r written in reverse order. Then, using Proposition 2, for transposed matrices ($X_r^\top Q_r^\top = C^\top X_r^\top$), we get that if z is a right eigenvector of $C^\top$, then $z = X_r^\top x$, where x is an eigenvector of Q_r and $X_r^\top$ is the transposed Krylov matrix. This allows to relate the vector of the number of quadrangulations with the characteristic polynomials. The vector $(q_0(\lambda), 2q_0(\lambda) - q_1(\lambda), 7q_0(\lambda) - 4q_1(\lambda) + q_2(\lambda), \ldots)$, obtained by using Corollary 2, equals the vector $(1, \lambda, \lambda^2, \ldots)$.

$$\sum_{i=1}^{r+1} v_i^{r+1} q_{i-1}(\lambda)(-1)^{i+1} = \lambda^r$$

If λ is an eigenvalue of the matrix Q_r, then $q_r(\lambda) = 0$ and we get the following proposition:

Proposition 3 *Let v^{r+1} be the vector that counts the number of quadrangulations on $r+1$ nodes, $q_i(\lambda)$ the characteristic polynomial of Q_i and λ being an eigenvalue of Q_r, then*

$$\sum_{i=1}^{r} v_i^{r+1} q_{i-1}(\lambda)(-1)^{i+1} = \lambda^r$$

We also observe that

$$q_r(\lambda) = \left(\lambda^r + \sum_{i=1}^{r} v_i^{r+1} q_{i-1}(\lambda)(-1)^i\right)(-1)^r$$

More properties concerning the eigenvalues of the production matrix Q_r are stated in the following propositions. We omit some technical straight-forward proofs in this section.

The largest eigenvalue of Q_r can be estimated in several ways, for instance using the power method. It is well known that the largest eigenvalue of the production matrix Q_r tends towards $\frac{27}{4}$ as r tends towards infinity [40]. To also estimate the other eigenvalues, one approach is to use the inverse power method. The idea of the method is that

$$\left((Q_r - \mu I_r)^{-1}\right)^r x$$

tends towards an eigenvector of $(Q_r - \mu I_r)^{-1}$ as r tends towards infinity. Thus, the key observation is that if λ is an eigenvalue of Q_r, then an eigenvalue of

$$(Q_r - \mu I_r)^{-1}$$

is given by $\frac{1}{\lambda-\mu}$ if $\mu \neq \lambda$.

Proposition 4 *Let Q_r be the $r \times r$ production matrix for quadrangulations, I_r the $r \times r$ identity matrix and $q_r(\lambda)$ be the characteristic polynomial of Q_r, then*

$$(Q_r - \mu I_r)^{-1} = \sum_{k=0}^{r} Q_r^k \left(\frac{\sum_{j=k}^{r} \binom{2j+4}{r-j-1}(-1)^j \mu^{j-k}}{q_r(\mu)}\right) \tag{3.5}$$

for $\mu \in \mathbb{R}$ not an eigenvalue of Q_r.

Proof: Using Cayley-Hamilton's theorem, we can write $(Q_r - \mu I_r)^{-1}$ as a sum of powers of Q_r. We have

$$q_r(\lambda + \mu) = \sum_{k=0}^{r} \binom{2k+2}{r-k}(\lambda+\mu)^k(-1)^k = \sum_{k=0}^{r} \binom{2k+2}{r-k} \sum_{j=0}^{k} \binom{k}{j} \lambda^j \mu^{k-j}(-1)^k$$

After some calculations and simplification, this gives the desired formula. □

Let $g(x)$ be a polynomial, then the eigenvalues of $g(Q_r)$ are given by $g(\lambda)$, where λ is an eigenvalue of Q_r. This implies that the eigenvalues λ of Q_r satisfy the following:

Proposition 5 *Let $q_r(\lambda)$ be the characteristic polynomial of the production matrix Q_r, λ an eigenvalue of Q_r and $\mu \in \mathbb{R}$ not an eigenvalue, then:*

$$\sum_{k=0}^{r} \lambda^k \left(\frac{\sum_{j=k}^{r} \binom{2j+4}{r-j-1}(-1)^j \mu^{j-k}}{q_r(\mu)}\right) = \frac{1}{\lambda - \mu} \tag{3.6}$$

The problem in applying the inverse power method by multiplying with the vector of quadrangulations is now that we only get a new vector $v^{\ell+1} = Q_r \cdot v^{\ell}$, for $\ell < r$, which counts the number of quadrangulations, but we would also need that ℓ can be greater than r, which is not guaranteed here.

The adjugate matrix $Adj(A)$ (also known as the transposed matrix of the cofactor matrix) of a matrix A satisfies $A \cdot Adj(A) = \det(A)$. It is known that if λ is an eigenvalue of the $n \times n$ matrix A, then the columns of $Adj(A - \lambda I_n)$ are eigenvectors of A [58]. We calculate $Adj(Q_r - \mu I_r)$, using Cayley-Hamilton's theorem.

Proposition 6 *Let Q_r be the $r \times r$ production matrix of quadrangulations and I_r the $r \times r$ identity matrix, then*

$$\begin{aligned} Adj(Q_r - \mu I_r) &= \sum_{k=1}^{r} (Q_r - \mu I_r)^{k-1} \sum_{j=k}^{r} \binom{2j+2}{r-j} \binom{j}{k} \mu^{j-k} (-1)^j = \\ &= \sum_{k=0}^{r-1} Q_r^k \sum_{\ell=k}^{r} \sum_{j=\ell+1}^{r} \binom{\ell}{k} \binom{2j+2}{r-j} \binom{j}{\ell+1} \mu^{j-k-1} (-1)^{j+\ell-k} \end{aligned}$$

We know that the first column of Q_r^k is a vector containing the number of quadrangulations on k 4-gons. Hence, we can express an eigenvector of Q_r in terms of the entry v_k^r and λ. Such an eigenvector w has the following entries: (for $1 \leq i \leq r$)

Corollary 4 *Let w be an eigenvector of Q_r, then for λ and eigenvalue of Q_r.*

$$w_{r,i} = \sum_{k=0}^{r-1} v_i^{k+1} \sum_{\ell=k}^{r} \sum_{j=\ell+1}^{r} \binom{\ell}{k} \binom{2j+2}{r-j} \binom{j}{\ell+1} \lambda^{j-k-1} (-1)^{j+\ell-k}$$

Proposition 7 *Let $q_i(\lambda)$ be the characteristic polynomial of the production matrix Q_i, then if $x \neq y$:*

$$\sum_{i=1}^{r} q_{r-i}(x) q_{i-1}(y) = \frac{q_r(y) - q_r(x)}{x - y}$$

Note that $q_0(x) = 1$. Then we can obtain the following result from Proposition 7

Corollary 5 *Let $q_i(\lambda)$ be the characteristic polynomial of the production matrix Q_i, then if $y \to x$, we obtain*

$$\sum_{i=1}^{r} q_{r-i}(x) q_{i-1}(x) = -q_r'(x)$$

If we use Proposition 7 with an eigenvalue λ of Q_r and some other value $\mu \in \mathbb{R}$, we get

$$\sum_{i=1}^{r} q_{r-i}(\lambda)q_{i-1}(\mu) = \frac{q_r(\mu)}{\lambda - \mu}$$

We also have, for $r \geq 2$,

$$\sum_{i=1}^{r} q_{r-i}(\lambda)q_{i-1}(\mu) = \sum_{k=1}^{r-1}\sum_{i=0}^{k} \binom{2k+4}{r-k-1}(-1)^k \lambda^i \mu^{k-i}$$

3.2 Pentangulations

In the second part of this chapter we are going to develop almost the same results as for quadrangulations, but for the class of 2-connected graphs whose internal faces are 5-gons.

The proofs of the theorems stated in this section are not written down here, all the proofs are developed in Section 3.3 for k-angulations.

3.2.1 Production matrix

An analogous process as for generating a production matrix for quadrangulations is made for finding a production matrix for the class of 2-connected graphs all whose internal faces are pentagons, i.e. the class of pentangulations. In this case, the production matrix is devised by using a surjective mapping of the graphs on $(r+1)$ 5-gons to the graphs on r 5-gons. So, the production matrix P_r of pentangulations must fulfill the following system:

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i+1} = P_r \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i}$$

where d_j^i is the number of pentangulations with i 5-gons and root vertex of degree $j-1$, following the same definition of degree as for quadrangulations: number of incident edges to the root vertex, minus 2.

Before defining the production matrix P_r, we must say that the dimension of the matrix is the number r of 5-gons. Note that this problem only makes sense if the number of nodes is $n = 2 + 3r, r \in \mathbb{N}$, otherwise, one of the faces would not be a 5-gon, because we have less vertices to make the pentangulation. That is the reason why we count the number of pentangulations with r 5-gons instead of the number of pentangulations with n vertices.

Using an analogous procedure as for the entries of the production matrix Q_r of quadrangulations, we obtain the entries of our production matrix P_r of pentangulations. To the best of our knowledge, this is the first time a production matrix for pentangulations is derived. However, generating trees for dissections into k-gons are known [2], but they are not stated in matrix form.

Theorem 12 *The following $r \times r$ matrix P_r is a production matrix for pentangulations of point sets in convex position.*

$$P_r = \begin{pmatrix} 3 & 6 & 10 & 15 & 21 & \cdots & \binom{r+2}{2} \\ 1 & 3 & 6 & 10 & 15 & \cdots & \binom{r+1}{2} \\ 0 & 1 & 3 & 6 & 10 & \cdots & \binom{r}{2} \\ 0 & 0 & 1 & 3 & 6 & \cdots & \binom{r-1}{2} \\ 0 & 0 & 0 & 1 & 3 & \cdots & \binom{r-2}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3 \end{pmatrix}$$

Proof: Following the notion of the flip operation defined in Section 3.1.1, we want to know the number of graphs with root vertex of degree $j-1$ on $r+1$ 5-gons, from the knowledge about v^r. Assume we are given a pentangulation $\sigma \in \mathfrak{P}_r$, and the vertex p_{3r+2} has degree t. First add three new vertices p_{3r+3}, p_{3r+4} and p_{3r+5} to the pentangulation by creating a new face with vertices $p_1, p_{3r+2}, p_{3r+3}, p_{3r+4}$ and p_{3r+5}. This gives a pentangulation $\sigma' \in \mathfrak{P}_{r+1}$. If we want $d(p_{3r+5}) = 0$ in σ', we have the possibility of flippling some of the t edges incident to p_{3r+2} in σ' except the ones joining it with p_{3r+1} and p_{3r+3}, such that they are then incident to p_{3r+3} or p_{3r+4}, i.e. replacing an edge $p_{3r+2}p_i$ either by the edge $p_{3r+3}p_{i+1}$ or by the edge $p_{3r+4}p_{i+2}$ for all p_i adjacent to p_{3r+2} in σ and obtaining $t+1$ pentangulations. In general, if we want $d(p_{3r+5}) = m,\ m \leq t-1$, we have to flip m edges from p_{3r+2} to p_{3r+5}, and the remaining of the $t-m$ edges of p_{3r+5}, we can leave them incident to p_{3r+2} or flip a subset of them from p_{3r+2} to p_{3r+3} and p_{3r+4}. This leads us to the production matrix for pentangulations P_r. $\square$

Let v^i be the vector that counts the number of pentangulations with i 5-gons and $n = 3r+2$ vertices whose j-th entry is the number of pentangulations where the root vertex has degree $j-1$. It is clear that v^i is the first column of a power of P_r for $i < r$, and the sum of the elements of v^i is the number of pentangulations of a set of $3i+2, i \in \mathbb{N}$ points in convex position. The first vectors are:

$$v^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad v^2 = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad v^3 = \begin{pmatrix} 15 \\ 6 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad v^4 = \begin{pmatrix} 91 \\ 39 \\ 9 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

Theorem 13 *Let v^r be the vector that counts the number of pentangulations with r 5-gons and $3r+2$ vertices, then the j-th entry of such vector, $\forall j = 1, \ldots, n-1$ is:*

$$v^r_j = \frac{j}{r}\binom{4r-j-1}{r-j}$$

If we sum up all the entries of the vector v^r, we obtain the sequence $\{1, 4, 22, 140, 969, \ldots\}$, given by the formula $\frac{1}{4r+1}\binom{4r+1}{r}$. This sequence can be found in the On-Line Encyclopedia of Integer Sequences [41] as Sequence $A002293$, and it also enumerates rooted quartic trees (i.e. trees with one root node, and where each node has either zero or four branches descending from it) with $n \geq 1$ vertices.

As mentioned in the previous section, the numbers of the form $\frac{r}{pn+1}\binom{pn+1}{n}$ are called Pfaff-Fuss-Catalan numbers.

In the previous section, we used a bijection between quadrangulations and ternary trees to verify that the formula $\frac{1}{3r+1}\binom{3r+1}{r}$ also gives the number of quadrangulations. Analogously, there also exists a bijection between quartic trees and pentangulations. This bijection is done in the same way to transform any given d-ary tree into a $(d+1)$-angulation: each internal node of the rooted d-ary tree corresponds to a face in the $(d+1)$-angulation and the leafs of the rooted tree corresponds to the edges of the boundary on the $(d+1)$-angulation. So, using this result, we can check that the formula $\frac{1}{4r+1}\binom{4r+1}{r}$ provides the number of pentangulations with $3r+2, r \in \mathbb{N}$ vertices.

3.2.2 Characteristic polynomial

In the following theorem we propose another new result in which the formula of the characteristic polynomial $p_r(\lambda)$ of the production matrix of pentangulations P_r has been calculated, satisfying a recurrence relation in terms of the previous terms $p_j(\lambda)$, for $j < r$.

The sequence $\{p_r(\lambda)\}$ starts with

$$
\begin{array}{r|c}
r=1 & -\lambda+3 \\
r=2 & \lambda^2-6\lambda+3 \\
r=3 & -\lambda^3+9\lambda^2-15\lambda+1 \\
r=4 & \lambda^4-12\lambda^3+36\lambda^2-20\lambda \\
r=5 & -\lambda^5-15\lambda^4-66\lambda^3+84\lambda^2-15\lambda \\
r=6 & \lambda^6-18\lambda^5+105\lambda^4-220\lambda^3+126\lambda^2-6\lambda
\end{array}
$$

Corollary 6 *The characteristic polynomial $p_r(\lambda)$ of the production matrix of pentangulations P_r satisfies the recurrence relation:*

$$p_r(\lambda) = (3-\lambda)p_{r-1}(\lambda) - \sum_{i=2}^{r}(-1)^i\binom{i+2}{2}p_{r-i}(\lambda)$$

The previous corollary follows from Theorem 6 in Section 2.

The following theorem gives the solution of the recurrence relation given by Corollary 6 for the characteristic polynomial $p_r(\lambda)$.

Theorem 14 *The solution of the recurrence relation*

$$p_r(\lambda) = (3-\lambda)p_{r-1}(\lambda) - \sum_{i=2}^{r}(-1)^i\binom{i+2}{2}p_{r-i}(\lambda)$$

with initial value $p_0(\lambda) = 1$ *is*

$$p_r(\lambda) = \sum_{k=0}^{r}\binom{3k+3}{r-k}\lambda^k(-1)^k$$

Having an expression for the characteristic polynomial allows us to apply Cayley-Hamilton's theorem (see Theorem 4), which in this case tells us that $\sum_{j=0}^{r}\binom{3j+3}{r-j}(-1)^j(P_r)^j = \mathbf{0}_r$ where $\mathbf{0}_r$ is the $r \times r$ zero matrix. This gives a relation among the number of pentangulations with root vertex of degree k. For example, for $k = 0$ (entry $(1,1)$ of P_r) we obtain the following relation:

Proposition 8 *Let* $r \geq 3$, *then:*

$$\sum_{j=0}^{r}\frac{1}{4j+1}\binom{4j+1}{j}\binom{3j+3}{r-j}(-1)^j = 0$$

3.3 k-angulations

Finally, we generalize all the previous results for the class of k-angulations.

3.3.1 Production matrix

In general, if we want to know the number of 2-connected geometric graphs on r k-gons, we obtain the following theorem.

Theorem 15 *The following* $r \times r$ *matrix* K_r *is a production matrix for* k*-angulations of point sets in convex position.*

$$\begin{pmatrix}
\binom{k-2}{k-3} & \binom{k-1}{k-3} & \binom{k}{k-3} & \binom{k+1}{k-3} & \binom{k+2}{k-3} & \cdots & \binom{r+k-3}{k-3} \\
1 & \binom{k-2}{k-3} & \binom{k-1}{k-3} & \binom{k}{k-3} & \binom{k+1}{k-3} & \cdots & \binom{r+k-4}{k-3} \\
0 & 1 & \binom{k-2}{k-3} & \binom{k-1}{k-3} & \binom{k}{k-3} & \cdots & \binom{r+k-5}{k-3} \\
0 & 0 & 1 & \binom{k-2}{k-3} & \binom{k-1}{k-3} & \cdots & \binom{r+k-6}{k-3} \\
0 & 0 & 0 & 1 & \binom{k-2}{k-3} & \cdots & \binom{r+k-7}{k-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \binom{k-2}{k-3}
\end{pmatrix}$$

The matrix has been derived following the same process as for the two previous production matrices Q_r and P_r, following the flip operation in the generating tree of k-angulations. And as we can see, the elements of each row of the matrix K_r of a k-angulation correspond to the $k-2$ diagonal of the Pascal triangle.

Now, we want to know the number of k-angulations on i k-gons with root vertex of degree $j-1$, and point sets in convex position, where the degree of the root vertex is defined as the number of incident edges to the root vertex, minus 2. This number is denoted by v_j^i and is obtained with the following theorem.

Theorem 16 *Let v^r be the vector that counts the number of k-angulations on r k-gons and $2+(k-2)r$ vertices, then the j-th entry of such vector, $\forall j = 1, \ldots, n-1$ is:*

$$v_j^r = \frac{j}{r}\binom{(k-1)r-j-1}{r-j}$$

Proof: The matrix K_r of k-angulations has a corresponding Riordan Array. Using notation as in [38], its A-sequence is $\{1, \binom{k-2}{k-3}, \binom{k-1}{k-3}, \binom{k}{k-3}, \binom{k+1}{k-3}, \ldots\}$ with generating function $A(t) = \frac{1}{(1-t)^{k-2}}$. As we have $d(t) = h(t)$, Theorem 3 states (after a shift of indices) that $v_j^r = \frac{j}{r}[t^{r-j}]A(t)^r$. Using the well-known identity (3.4), we expand

$$A(t)^r = \left(\frac{1}{1-t}\right)^{(k-2)r} = \sum_{\ell=0}^{\infty}\binom{(k-2)r+\ell-1}{\ell}t^\ell$$

It remains to determine the coefficient of t^{r-j}. We set $r-j=\ell$. We thus arrive at the claimed formula

$$v_j^r = \frac{j}{r}[t^{r-j}]A(t)^r = \frac{j}{r}\binom{(k-2)r+r-j-1}{r-j} = \frac{j}{r}\binom{(k-1)r-j-1}{r-j}$$

□

3.3.2 Characteristic polynomial

As we have done in the previous sections for $k=4$ and $k=5$, we now proceed to establish a new recurrence relation for a generic k satisfied by the characteristic polynomial. The proof of the theorem is given by applying Theorem 6 with $a_{-1}=1$ and $a_{i-1}=i+1$.

Corollary 7 *The characteristic polynomial $k_r(\lambda)$ of the matrix K_r satisfies the recurrence relation*

$$k_r(\lambda) = \left(\binom{k-2}{k-3}-\lambda\right)k_{r-1}(\lambda) + \sum_{i=2}^{r}(-1)^{i+1}\binom{k+i-3}{k-3}k_{r-i}(\lambda)$$

Next, we want to obtain the solution of the recurrence relation given by Corollary 7. However, before we state the formula of the solution, we need a previous result.

Lemma 1 *For any given $t, m, n \in \mathbb{N}$*

$$\sum_{j=0}^{n}\binom{j+(m-1)}{m-1}\binom{m(t+1)}{n-j-t}(-1)^j = \binom{mt}{n-t}$$

Proof: We use induction on t. The base cases for $t = 0, t = 1, t = 2$ and $t = 3$ are easily verified. Let then $t \geq 4$ and assume the lemma holds for all $0 \leq i \leq t-1$. By Vandermonde's Convolution, we know that

$$\binom{mt}{n-t} = \binom{m + m(t-1)}{n-t} = \sum_{s=0}^{n-t} \binom{m}{s}\binom{m(t-1)}{(n-t)-s} = \sum_{s=0}^{m} \binom{m}{s}\binom{m(t-1)}{(n-t)-s} =$$
$$= \sum_{s=0}^{m} \binom{m}{s}\binom{m(t-1)}{n-(s+1)-(t-1)}$$

So, by induction hypothesis, this is equal to

$$= \sum_{s=0}^{m} \binom{m}{s} \sum_{j=0}^{n-(s+1)} \binom{j+(m-1)}{m-1}\binom{mt}{n-(s+1)-j-(t-1)}(-1)^j$$
$$= \sum_{j=0}^{n-(m+1)} \binom{j+(m-1)}{m-1} \sum_{s=0}^{m} \binom{m}{s}\binom{mt}{n-(s+1)-j-(t-1)}(-1)^j$$
$$= \sum_{j=0}^{n-(m+1)} \binom{j+(m-1)}{m-1}\binom{m+mt}{n-j-t}(-1)^j = \sum_{j=0}^{n} \binom{j+(m-1)}{m-1}\binom{m(t+1)}{n-j-t}(-1)^j$$

□

Now, we have all the tools needed to solve the recurrence relation of the characteristic polynomial of Corollary 7.

Theorem 17 *The solution of the recurrence relation*

$$k_r(\lambda) = \left(\binom{k-2}{k-3} - \lambda\right) k_{r-1}(\lambda) - \sum_{i=2}^{r} (-1)^i \binom{k+i-3}{k-3} k_{r-i}(\lambda)$$

with initial value $k_0(\lambda) = 1$ *is*

$$k_r = \sum_{\ell=0}^{r} (-1)^\ell \binom{(k-2)(\ell+1)}{r-\ell} \lambda^\ell$$

Proof: We now use induction on r to prove the theorem. The base cases are easily verified for $r = 0$ or $r = 1$. So, assume the theorem holds for all $0 \leq i \leq r-1$. Consider then $k_r(\lambda)$. By induction,

$$k_r(\lambda) = \left(\binom{k-2}{k-3} - \lambda\right) \sum_{\ell=0}^{r-1} \binom{(k-2)(\ell+1)}{r-1-\ell} \lambda^\ell (-1)^\ell -$$
$$- \sum_{i=2}^{r} \binom{i+k-3}{k-3} \sum_{\ell=0}^{r-i} \binom{(k-2)(\ell+1)}{r-i-\ell} \lambda^\ell (-1)^{\ell+i}$$

We rewrite

$$A = \sum_{i=2}^{r} \binom{i+(k-3)}{(k-3)} (-1)^i \sum_{\ell=0}^{r-i} \binom{(k-2)(\ell+1)}{r-i-\ell} \lambda^\ell (-1)^\ell$$

in the form $\sum_{t=0}^{r} \lambda^t c_t$, where c_t is the coefficient of λ^t and get

$$A = \sum_{t=0}^{r} \lambda^t \sum_{i=2}^{r} \binom{i+(k-3)}{(k-3)} \binom{(k-2)(t+1)}{r-i-t} (-1)^{i+t}$$

This further equals, by Lemma 1 above,

$$= \sum_{t=0}^{r} \lambda^t (-1)^t \left(\binom{(k-2)t}{r-t} - \binom{(k-2)(t+1)}{r-t} + \binom{k-2}{k-3} \binom{(k-2)(t+1)}{r-t-1} \right)$$

Then,

$$\begin{aligned} k_r(\lambda) = \left(\binom{k-2}{k-3} - \lambda \right) \sum_{\ell=0}^{r-1} \binom{(k-2)(\ell+1)}{r-1-\ell} \lambda^\ell (-1)^\ell - \sum_{t=0}^{r} \lambda^t (-1)^t \left(\binom{(k-2)t}{r-t} - \right. \\ \left. - \binom{(k-2)(t+1)}{r-t} + \binom{k-2}{k-3} \binom{(k-2)(t+1)}{n-t-1} \right) \\ = \sum_{\ell=0}^{r-1} \binom{k-2}{k-3} \binom{(k-2)(\ell+1)}{r-1-\ell} \lambda^\ell (-1)^\ell - \sum_{\ell=0}^{r-1} \binom{(k-2)(\ell+1)}{r-1-\ell} \lambda^{\ell+1} (-1)^\ell \\ - \sum_{\ell=0}^{r} \lambda^\ell (-1)^\ell \left(\binom{(k-2)\ell}{r-\ell} - \binom{(k-2)(\ell+1)}{r-\ell} + \binom{k-2}{k-3} \binom{(k-2)(\ell+1)}{r-\ell-1} \right) \end{aligned}$$

We make the change of variables $\ell+1 = s$ in the second summation and get:

$$\begin{aligned} = \sum_{\ell=0}^{r} \binom{k-2}{k-3} \binom{(k-2)(\ell+1)}{r-1-\ell} \lambda^\ell (-1)^\ell - \sum_{s=1}^{r} \binom{(k-2)s}{r-s} \lambda^s (-1)^{s-1} \\ - \sum_{\ell=0}^{r} \lambda^\ell (-1)^\ell \left(\binom{(k-2)\ell}{r-\ell} - \binom{(k-2)(\ell+1)}{r-\ell} + \binom{k-2}{k-3} \binom{(k-2)(\ell+1)}{r-\ell-1} \right) \\ = \sum_{\ell=0}^{r} \binom{k-2}{k-3} \binom{(k-2)(\ell+1)}{r-1-\ell} \lambda^\ell (-1)^\ell + \sum_{s=1}^{r} \binom{(k-2)s}{r-s} \lambda^s (-1)^s \\ - \sum_{\ell=0}^{r} \lambda^\ell (-1)^\ell \binom{(k-2)\ell}{r-\ell} + \sum_{\ell=0}^{r} \lambda^\ell (-1)^\ell \binom{(k-2)(\ell+1)}{r-\ell} - \\ - \sum_{\ell=0}^{r} \binom{k-2}{k-3} \binom{(k-2)(\ell+1)}{r-\ell-1} \lambda^\ell (-1)^\ell = \sum_{\ell=0}^{r} \lambda^\ell (-1)^\ell \binom{(k-2)(\ell+1)}{r-\ell} \end{aligned}$$

$\square$

3.3.3 Eigenvectors of K_r

In this section we give a formula for the entries of an eigenvector x associated to the production matrix K_r and the eigenvalue λ.

Corollary 8 *Let $x = (x_{r-1}, x_{r-2}, \ldots, x_0)$ be an eigenvector associated to an eigenvalue λ of the matrix K_r. Then the entries of the vector x are of the form:*

$$x_i = (-1)^i k_i(\lambda) x_0$$

where $k_i(\lambda)$ is the characteristic polynomial of K_i.

The proof of this corollary is given by Theorem 6 taking $a_{-1} = 1$.

3.3.4 Other properties

As we have said for quadrangulations, there exist matrices that play an important role while calculating results about the characteristic polynomial of a matrix. In this section, we generalize some of the results given in Section 3.1.5. The Krylov matrix X_r associated to the production matrix K_r of k-angulations and the vector $b = (1, 0, 0, 0, \ldots, 0)^\top$ is featured below:

$$X_r = \begin{pmatrix} 1 & k-2 & \frac{1}{3}\binom{3k-5}{2} & \frac{1}{4}\binom{4k-6}{3} & \frac{1}{5}\binom{5k-7}{4} & \cdots & v_1^r \\ 0 & 1 & 2(k-2) & \frac{1}{2}\binom{4k-7}{2} & \frac{2}{5}\binom{5k-8}{3} & \cdots & v_2^r \\ 0 & 0 & 1 & 3(k-2) & \frac{3}{5}\binom{5k-9}{2} & \cdots & v_3^r \\ 0 & 0 & 0 & 1 & 4(k-2) & \cdots & v_4^r \\ 0 & 0 & 0 & 0 & 1 & \cdots & v_5^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where v_j^i is the number of k-angulations on i k-gons and root vertex of degree $j-1$.

In addition, since the solution of the characteristic polynomial of K_r is given by Theorem 17 as

$$k_r(\lambda) = \sum_{\ell=0}^{r} \binom{(k-2)(\ell+1)}{r-\ell} (-1)^\ell \lambda^\ell$$

we obtain the following companion matrix:

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & -\binom{k-2}{r} \\ 1 & 0 & 0 & 0 & \cdots & 0 & \binom{2(k-2)}{r-1} \\ 0 & 1 & 0 & 0 & \cdots & 0 & -\binom{3(k-2)}{r-2} \\ 0 & 0 & 1 & 0 & \cdots & 0 & \binom{4(k-2)}{r-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & (-1)^{r-1}\binom{(k-2)(r-1)}{2} \\ 0 & 0 & 0 & 0 & \cdots & 1 & (-1)^r\binom{(k-2)r}{1} \end{pmatrix}$$

By Proposition 2, we can calculate the eigenvectors of the companion matrix C by calculating the inverse of the matrix X_r and multiplying it by the eigenvector of the matrix K_r given by Theorem 8.

Theorem 18 *Let X_r be the Krylov matrix associated to the production matrix K_r, then the inverse of X_r has the following form:*

$$X_r^{-1} = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & \cdots & x_{1,r} \\ 0 & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & \cdots & x_{2,r} \\ 0 & 0 & x_{3,3} & x_{3,4} & x_{3,5} & \cdots & x_{3,r} \\ 0 & 0 & 0 & x_{4,4} & x_{4,5} & \cdots & x_{4,r} \\ 0 & 0 & 0 & 0 & x_{5,5} & \cdots & x_{5,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & x_{r,r} \end{pmatrix}$$

with entries $x_{i,j} = (-1)^{i+j}\binom{(k-2)i}{j-i}$

Proof: The proof is based on Riordan Arrays, following results in [36] and [52]. According to [36], every Riordan Array induces the two-parameters basic identity:

$$\sum_{j=s}^{n} d_{n,j} d^*_{j,s} = \delta_{n,s}$$

where $d_{i,j}$ is the generic element of a Riordan Array, the superscripted asterisk denotes the following quantity related to the inverse Riordan Arrays

$$d^*_{r,s} = \frac{1}{r}[t^{r-s}]\left(\frac{s}{d(t)} - \frac{td'(t)}{d(t)^2}\right)\left(\frac{1}{h(t)}\right)^r$$

and $\delta_{n,s}$ is the Kronecker delta.

In our case, we prove that $\sum_{h=s}^{r} v_s^h (-1)^{h-s}\binom{(k-2)h}{r-h} = \delta_{r,s}$, where v^h is the vector that counts the number of k-angulations on h points. This identity can be seen as if we have to proof that $X_r \cdot X_r^{-1} = I_r$, where I_r is the $r \times r$ identity matrix.

Theorem 2.1 in [52] gives a formula for calculating the elements of a given Riordan Array. If we consider the binomial coefficient $\binom{(k-2)s}{r-s}$ we have the infinite array given by the Riordan Array $D = (d(t), h(t)) = (1, (1+t)^{k-2})$. Hence, we have to prove that $d^*_{r,s} = v_s^r(-1)^{r-s}$. We can prove this result with the formula given by [36, Theorem 2.2]:

$$d^*_{r,s} = \frac{1}{r}[t^{r-s}]\left(\frac{s}{d(t)} - \frac{td'(t)}{d(t)^2}\right)\left(\frac{1}{h(t)}\right)^r = \frac{1}{r}[t^{r-s}]s\left(\frac{1}{1+t}\right)^{(k-2)r} =$$

$$= \frac{s}{r}[t^{r-s}]\sum_{\ell=0}^{\infty}\binom{(k-2)r+\ell-1}{\ell}(-t)^{\ell} = \frac{s}{r}\binom{(k-1)r-s-1}{r-s}(-1)^{r-s} = v_s^r(-1)^{r-s}$$

□

As we can observe, the i-th row of the inverse of the Krylov matrix corresponds to the $(ki+1)$-th row of the Pascal triangle (see Figure 1.6) with some entries multiplied by -1.

We thus arrive to the following combinatorial identity:

Corollary 9 *Let $k \geq 3$, and $s, r \in \mathbb{N}$, then*

$$\sum_{h=s}^{r} \frac{s}{h} \binom{(k-1)h-s-1}{h-s} \binom{(k-2)h}{r-h} (-1)^{h-s} = \delta_{r,s}$$

where $\delta_{r,s}$ is the Kronecker delta.

Chapter 4

Geometric graphs

In this Chapter, we work with graphs whose vertices are represented by distinct points $\{g_1, \ldots, g_n\}$ in convex position in the plane and whose edges are drawn as straight-line segments. Moreover, we focus on geometric graphs with the vertices in convex position and without edge crossing. Figures 4.1, 4.2 and 4.3 show several geometric graphs.

4.1 Production matrix

In previous papers such as [29], a production matrix G_n for the number of graphs with given root vertex degree was given. The matrix G_n is defined as follows:

$$G_n = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 3 & 3 & 3 & \cdots & 3 \\ 0 & 2 & 4 & 4 & 4 & \cdots & 4 \\ 0 & 0 & 2 & 4 & 4 & \cdots & 4 \\ 0 & 0 & 0 & 2 & 4 & \cdots & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 4 \end{pmatrix} \tag{4.1}$$

In that case, the degree of the root vertex g_n is the number of edges incident to g_n. Also in that work, a formula for the number of geometric graphs is given. The formula of the k-th entry of the vector v^i that satisfies $v^{i+1} = G_n v^i$ for $i < n$ in [29] is

$$v_k^i = \frac{2^{i-2}}{i-1} \sum_{j=0}^{i-1} \binom{i-1}{j} \left(k \binom{i+j-k-2}{j-k} - (k+2) \binom{i+j-k-4}{j-k-2} \right) \tag{4.2}$$

where i is the number of vertices in which the root vertex has degree k.

However, there are other ways to define the degree. So, in this chapter, the degree of the root vertex is defined in a different way, based on visibility. Let $\{g_1, g_2, \ldots, g_n\}$ be the set of vertices of a geometric graph in convex position, then the degree of the root vertex of a geometric graph is the number of vertices visible from a vertex g_{n+1} inserted between g_1 and g_n in convex position, minus 2. Two vertices are visible if the line segment connecting them does not intersect the interior of any edge of the graph. For instance, if we have a look at the geometric graphs of Figure 4.1, we observe that in the right graph, the degree of the root vertex g_{12} for both definitions coincides, and it is 3. However, in the left graph, with the degree definition of [29], the root vertex g_{12} has degree 0, and with this new definition g_{12} has degree 3.

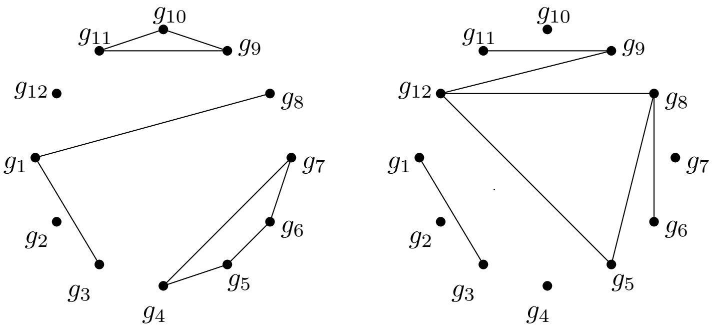

Figure 4.1: Geometric graphs on $n = 12$ vertices.

The first levels of the resulting generating tree of non-crossing geometric graphs are the following:

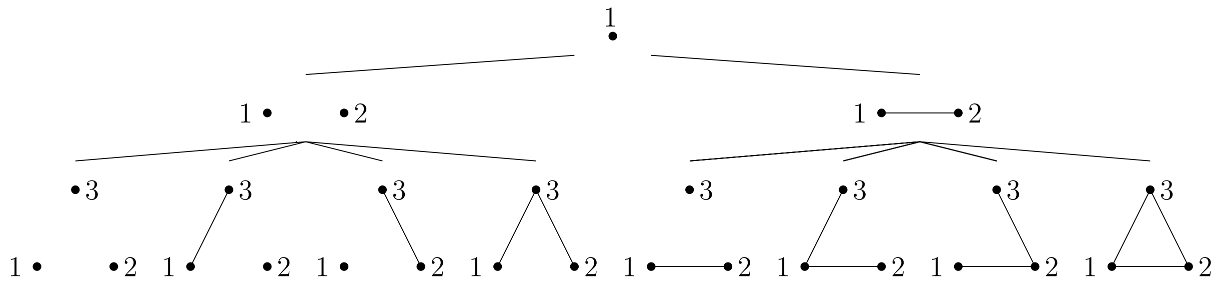

Figure 4.2: First levels of the tree of geometric graphs using visibility.

In general, for a given geometric graph, we generate its children by leaving the new root vertex isolated or connecting it to a subset of nodes visible from the new root vertex as in the following example:

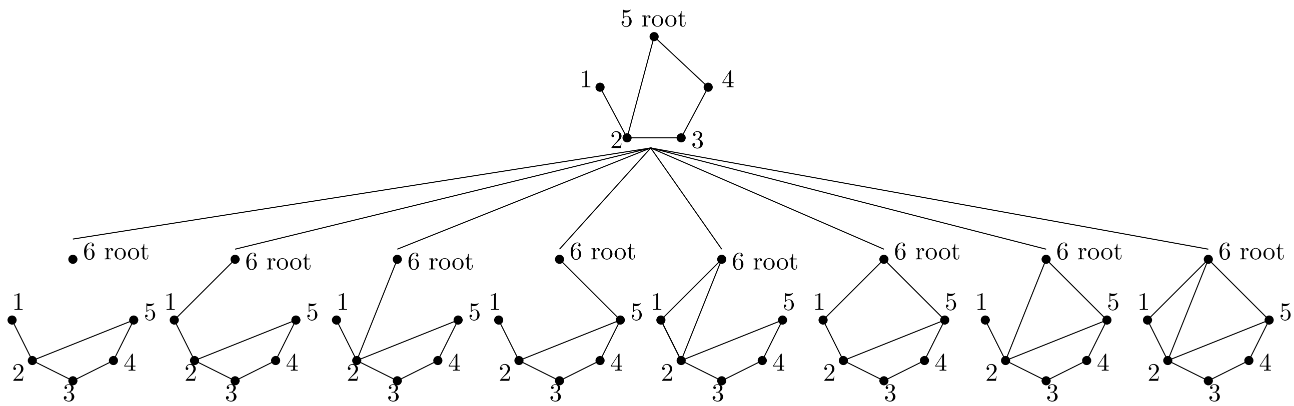

Figure 4.3: Children of a given geometric graph in the tree of geometric graphs.

Let $F(g_{i+1}) = \{g_1 = g^1_{i+1}, g^2_{i+1}, \dots, g^k_{i+1} = g_i\}, \quad k < i+1$ be the ordered sequence of the vertices visible from g_{i+1} inserted between g_1 and g_i. As we can see, the number of children of a given geometric graph with root vertex of degree $j-1$ is 2^{j+1}. And this is because we have $\binom{j+1}{0}$ ways of leaving g_{i+1} isolated, $\binom{j+1}{1}$ ways of adding an edge from g_{i+1} to one vertex of $F(g_{i+1})$, $\binom{j+1}{2}$ ways of adding two edges from g_{i+1} to a pair of vertices of $F(g_{i+1})$ and so on.

Theorem 19 *The following $n \times n$ matrix G_n is a production matrix for geometric graphs of point sets in convex position.*

$$G_n = \begin{pmatrix} 2 & 4 & 8 & 16 & 32 & \cdots & 2^n \\ 2 & 2 & 4 & 8 & 16 & \cdots & 2^{n-1} \\ 0 & 2 & 2 & 4 & 8 & \cdots & 2^{n-2} \\ 0 & 0 & 2 & 2 & 4 & \cdots & 2^{n-3} \\ 0 & 0 & 0 & 2 & 2 & \cdots & 2^{n-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

Proof: For v^{i+1}, consider a vertex g_{i+1} inserted between g_1 and g_i. Assume that the vector v^i, containing the number of geometric graphs for each possible visibility degree of g_i, is known. From the definition of the generating tree of geometric graphs, and the relation $v^{i+1} = G_n v^i$ we can determine the entries of G_n as follows:

- **First row**. The number of geometric graphs where g_{i+1} has degree 0 is equal to the number of all the graphs where g_i has degree $t, \quad t = 0, \dots, n-2$ when adding an edge from g_{i+1} to $g^1_{i+1} \in F(g_{i+1})$ and then adding a subset of edges from g_{i+1} to g^i_{i+1}, with $i > 1$. This gives a first row of powers of 2 in the matrix.

- **Second row**. The number of geometric graphs where g_{i+1} has degree 1, obtained from the graphs where g_i has degree 0 is equal to the two graphs leaving g_{i+1} isolated or connecting it with g_i, thus we get a 2 in the first column of the second row. As soon as g_i has degree at least 1, we have to add one edge from g_{i+1} to $g^2_{i+1} \in F(g_{i+1})$, and then we add a subset of edges from g_{i+1} to $g^i_{i+1} \in F(g_{i+1})$ for $i > 2$. Thus, the rest of the row is made of powers of 2.

- **Other rows**. The cases where $|F(g_{n+1})| > |F(g_n)| + 1$ are not possible, so we get a zero in these cases.
  The following rows are analogous, shifted by one column every time: in order for g_{i+1} to have degree t, one edge need to be added from g_{i+1} to $g^{t+1}_{i+1} \in F(g_{i+1})$, and then we can add a subset of edges from g_{i+1} to $g^s_{i+1} \in F(g_{i+1}), s \geq t+1$. So, we get a power of 2 for each entry. □

Let v^i be the vector of geometric graphs (that is, v^i is the first column of a power of G_n for $i < n$; the sum of the elements of v^i is the number of geometric graphs on i vertices). The first vectors are:

$$v^2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^3 = \begin{pmatrix} 4 \\ 4 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^4 = \begin{pmatrix} 24 \\ 16 \\ 8 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^5 = \begin{pmatrix} 176 \\ 112 \\ 48 \\ 16 \\ 0 \\ \vdots \end{pmatrix}$$

If we sum up all the entries of each vector v^i, we can verify that the sequence of numbers is the same as the result given in [29] and given by Sequence $A054726$ in the On-Line Encyclopedia of Integer Sequences [41] that counts the number of graphs on n nodes on a circle without crossing edges.

In the following theorem we give a formula to calculate the entries of the vector v^n of geometric graphs, different from (4.2). The difference comes from the definition of the degree of the root vertex. Recall that we define the degree of the root vertex g_n as the number of nodes visible from a vertex g_{n+1} inserted between the root vertex g_n and g_1, minus 2, whereas in the formula of Huemer et al. [29], they define the degree of the root vertex as the number of edges incident to it.

Theorem 20 *Let v^n be the vector that counts the number of geometric graphs on n vertices, then the k-th entry of such vector, $\forall k = 1, \ldots, n-1$ is:*

$$v_k^n = \frac{k}{n-1} 2^{n-1-k} \sum_{j=0}^{n-1} \binom{n-1}{j} \binom{n+j-k-2}{j-k} (-1)^{n-1-j} 2^j$$

Proof: The matrix G_n of geometric graphs has a corresponding Riordan Array. The second row of G_n gives the A-sequence $\{2, 2, 4, 8, 16, \ldots\}$ with generating function $A(t) = \frac{2-2t}{1-2t}$ and its Z-sequence is given by the first row of G_n and is $\{2, 4, 8, 16, 32, \ldots\}$ with generating function $Z(t) = \frac{2}{1-2t}$. We have $d_0 = 2$ (because use as starting vector v^2) and by the formula given by (1.3)

$$h(t) = A(th(t)) = \frac{2 - 2th(t)}{1 - 2th(t)}$$

We substitute $th(t) = w$. Then we get

$$w = t\frac{2 - 2w}{1 - 2w}$$

We define a function $\phi(w) = \frac{2-2w}{1-2w}$. Then $w = t\phi(w)$ and we can apply Lagrange inversion. But first, we calculate $d(t)$ also with the formula given by (1.3):

$$d(t) = \frac{d_0}{1 - tZ(th(t))} = \frac{2}{1 - t(\frac{2}{1-2th(t)})} = \frac{2 - 4th(t)}{1 - 2th(t) - 2t}$$

We substitute $th(t) = w$ and obtain

$$d(t) = \frac{h(t)(2-4w)}{h(t) - 2wh(t) - 2w} = \cdots = \frac{2-2w}{1-2w}$$

From this result and [38, Theorem 3.8] we see that $d(t) = h(t)$. Then, [38, Theorem 3.4] (i.e. Theorem 3 in this thesis) states that $d_{n,k} = \frac{k+1}{n+1}[t^{n-k}]A(t)^{n+1}$. It follows from this theorem together with Theorem 3.10, (and a shift of index) that the k-th entry of v^n is the coefficient of t^{n-k-1} in the Taylor expansion of $\frac{k}{n-1}(\frac{2-2t}{1-2t})^{n-1}$. Using the well-known identity (3.4), we expand

$$A(t)^{n-1} = \left(\frac{2-2t}{1-2t}\right)^{n-1} = 2^{n-1}\left(\frac{1}{1-2t} - \frac{t}{1-2t}\right)^{n-1}$$

And now, using the binomial formula for the algebraic expansion of powers of a binomial, we get the following result.

$$\begin{aligned}
A(t)^{n-1} &= 2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}\left(\frac{-t}{1-2t}\right)^{n-1-j}\left(\frac{1}{1-2t}\right)^{j} = \\
&= 2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}(-t)^{n-1-j}\left(\frac{1}{1-2t}\right)^{n-1} = 2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{n-1-j}t^{n-1-j}\left(\frac{1}{1-2t}\right)^{n-1} = \\
&= 2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{n-1-j}t^{n-1-j}\sum_{\ell=0}^{\infty}\binom{n-1+\ell-1}{\ell}(2t)^{\ell} = \\
&= 2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{n-1-j}\sum_{\ell=0}^{\infty}\binom{n-1+\ell-1}{\ell}2^{\ell}t^{n-1-j+\ell}
\end{aligned}$$

It remains to determine the coefficient of t^{n-k-1}. We set $n-k-1 = n-1-j+\ell$ and therefore $\ell = j-k$. We thus arrive at the claimed formula

$$\begin{aligned}
v_k^n &= \frac{k}{n-1}2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{n-1-j}\binom{n+j-k-2}{j-k}2^{j-k} \\
&= \frac{k}{n-1}(-2)^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^{-j}\binom{n+j-k-2}{j-k}2^{j-k}
\end{aligned}$$

□

4.2 Characteristic polynomial

As we have a new production matrix G_n different from the one previously known [29], we calculate a recurrence relation for the characteristic polynomial $g_n(\lambda)$ as well as its solution, given an initial condition. Then we proceed to calculate a formula for the eigenvectors of the production matrix G_n. So we obtain the following results. This first result follows from Theorem 6 with $a_{-1} = 2$ and $a_{i-1} = 2^i$.

The sequence $\{g_n(\lambda)\}$ starts with

$$\begin{array}{r|c} n=1 & -\lambda+2 \\ n=2 & \lambda^2-4\lambda-4 \\ n=3 & -\lambda^3+6\lambda^2+4\lambda+8 \\ n=4 & \lambda^4-8\lambda^3-16 \\ n=5 & -\lambda^5+10\lambda^4-8\lambda^3-16\lambda^2-16\lambda+32 \\ n=6 & \lambda^6-12\lambda^5+20\lambda^4+32\lambda^3+48\lambda^2+64\lambda-64 \end{array}$$

Corollary 10 *The characteristic polynomial $g_n(\lambda)$ of the matrix G_n satisfies the recurrence relation*

$$g_n(\lambda) = (2-\lambda)g_{n-1}(\lambda) + \sum_{i=2}^{n}(-1)^{i+1}2^{2i-1}g_{n-i}(\lambda)$$

Theorem 21 *The solution of the recurrence relation*

$$g_n(\lambda) = (2-\lambda)g_{n-1}(\lambda) + \sum_{i=2}^{n}(-1)^{i+1}2^{2i-1}g_{n-i}(\lambda)$$

with initial condition $g_0(\lambda) = 1$ is

$$g_n(\lambda) = \sum_{t=0}^{n}\sum_{k=0}^{n}(-1)^k\binom{k}{t}\binom{t+1}{n-k}2^{2n-t-k}\lambda^t$$

Proof: Consider the infinite matrix matrix

$$M = \begin{pmatrix} 2-\lambda & -8 & 32 & -128 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

let w_0 be the vector $(1,0,\ldots)^\top$, and let w_i be the vector whose first i entries are the first i characteristic polynomials $g_i(\lambda)$, and the remaining ones are zero. That is, $w_i = (g_i(\lambda), g_{i-1}(\lambda), g_{i-2}(\lambda), \ldots, g_1(\lambda), 0, \ldots)^\top$. Then $M \cdot w_i = w_{i+1}$. It follows that w_n is the first column of M^n. We now can use the Riordan Array approach. The Z-sequence is $\{2-\lambda, -8, 32, \ldots\}$ with generating function $Z(t) = 2-\lambda+\sum_{k=1}^{\infty}(-1)^k 2^{2k+1}t^k = 2-\lambda+2\sum_{k=1}^{\infty}(-1)^k(4t)^k = -\lambda+\frac{2}{1+4t}$. The A-sequence is $\{1,0,\ldots\}$ with generating function $A(t) = 1$. It follows that $h(t) = 1$ and

$$d(t) = \frac{1}{1-t(-\lambda+\frac{2}{1+4t})} = \frac{1}{1+t(2+\lambda(1+4t))} + \frac{4t}{1+t(2+\lambda(1+4t))} = d_1(t) + 4td_1(t)$$

where

$$d_1(t) = \frac{1}{1+t(2+\lambda(1+4t))}$$

Then,

$$d_{n,k} = [t^n]d_1(t)(th(t))^k + 4[t^{n-1}]d_1(t)(th(t))^k \tag{4.3}$$

If we set $z = 2 + \lambda(1 + 4t)$, then we know that

$$\frac{1}{1+z} = \sum_{k=0}^{\infty}(-1)^k z^k$$

Therefore

$$d_1(t) = \sum_{k=0}^{\infty} t^k(-1)^k(2 + \lambda(1 + 4t))^k$$

We apply the binomial theorem to $(2 + \lambda(1 + 4t))^k$. We then only have to take the coefficient of t^n in $d_1(t)$ for the left term of Equation (4.3) and the coefficient of t^{n-1} in $d_1(t)$ for the right term of Equation (4.3). This, after some short calculations, gives the formula

$$g_n(\lambda) = \sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^k\binom{k}{j}\binom{k-j}{n-k}2^{2(n-k)+j}\lambda^{k-j}+\sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^k\binom{k}{j}\binom{k-j}{n-k-1}2^{2(n-k)+j}\lambda^{k-j}$$

we finally rewrite $g_n(\lambda) = \sum_{t=0}^{n} c_t\lambda^t$, with c_t the coefficient of λ^t. Since $\lambda^t = \lambda^{k-j}$ we set $j = k - t$ and get

$$g_n(\lambda) = \sum_{t=0}^{n}\left(\sum_{k=0}^{n}(-1)^k\binom{k}{k-t}\binom{t}{n-k}2^{2n-t-k} + \sum_{k=0}^{n}(-1)^k\binom{k}{k-t}\binom{t}{n-k-1}2^{2n-t-k}\right)\lambda^t$$

After some calculations and taking into acount that $\binom{t}{n-k} + \binom{t}{n-k-1} = \binom{t+1}{n-k}$, and the symmetry identity for the binomial coefficient $\binom{k}{k-t}$, we finally get the formula

$$g_n(\lambda) = \sum_{t=0}^{n}\sum_{k=0}^{n}(-1)^k\binom{k}{t}\binom{t+1}{n-k}2^{2n-t-k}\lambda^t$$

□

4.3 Eigenvectors of G_n

So we will conclude this chapter with a property concerning the eigenvectors of the production matrix G_n. We give a formula for the entries of the eigenvector x associated to the production matrix G_n and the eigenvalue λ.

Corollary 11 *Let $x = (x_{n-1}, x_{n-2}, \ldots, x_0)$ be an eigenvector associated to an eigenvalue λ of the production matrix G_n. Then the entries of the vector x are of the form:*

$$x_i = \left(\frac{-1}{2}\right)^i g_i(\lambda)x_0$$

where $g_i(\lambda)$ is the characteristic polynomial of G_i.

The proof of the corollary follows from Theorem 6 with $a_{-1} = 2$.

Chapter 5

Connected graphs

A graph is said to be connected if there is a path between every pair of vertices. In our case, we treat connected graphs with vertices in convex position and without edge crossings. Figures 5.1 and 5.2 show several connected graphs.

5.1 Production matrix

As far as we know, there are not any previous results where the production matrix for connected graphs has been given, so all the results given in this chapter are new.

Also in this case, the degree of the root vertex is defined based on visibility. Let $\{c_1, c_2, \ldots, c_n\}$ be a set of points in convex position. Then the degree of the root vertex c_n is the number of vertices visible from a vertex c_{n+1} inserted in convex position between c_1 and c_n, minus 2. Two vertices are visible if the line segment connecting them does not intersect the interior of any edge of the graph. For this class of graphs, we have obtained a production matrix C_n that fulfills the following:

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i+1} = C_n \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i}$$

where d_j^i is the number of connected graphs on i vertices with root vertex of degree $j-1$.

For instance, if we have a look at the connected graphs of Figure 5.1, we observe that in the right graph, the degree of the root vertex c_{12} based on visibility coincides with the vertex degree based on incident edges (see [29]), and it is 3. However, in the left graph, with the vertex degree definition of [29], the root vertex c_{12} has degree 1, and with this new definition c_{12} has degree 2.

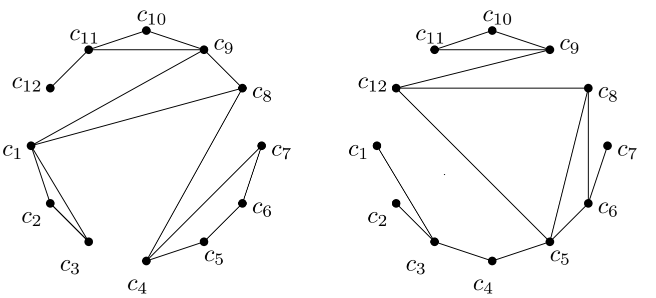

Figure 5.1: Connected graphs on $n = 12$ vertices.

In general, for a given connected graph, we generate its children by using the following mapping. Let $F(c_{i+1}) = \{c_1 = c_{i+1}^1, c_{i+1}^2, \ldots, c_{i+1}^t = c_i\}$, $t < i + 1$ be the ordered sequence of the vertices visible from a new vertex c_{i+1} inserted between c_1 and c_i. We obtain a connected graph by connecting c_{i+1} to all its visible vertices from c_k to c_j where $c_k, c_j \in F(c_{i+1})$ for each $k \leq j$, and removing a subset of the edges whose endpoints are between c_k and c_j in $F(c_{i+1})$, maintaining connectivity. We can see an example in the following figure:

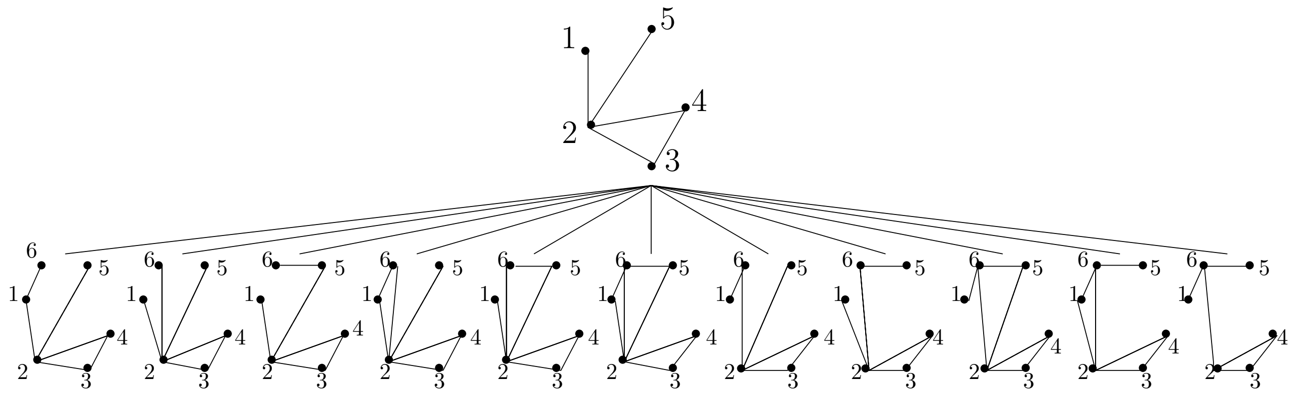

Figure 5.2: Children of a given connected graph in the tree of connected graphs.

Theorem 22 *The following $n \times n$ matrix C_n is a production matrix for non-crossing connected graphs of point sets in convex position.*

$$C_n = \begin{pmatrix} 3 & 7 & 15 & 31 & 63 & \cdots & 2^{n+1}-1 \\ 1 & 3 & 7 & 15 & 31 & \cdots & 2^n - 1 \\ 0 & 1 & 3 & 7 & 15 & \cdots & 2^{n-1}-1 \\ 0 & 0 & 1 & 3 & 7 & \cdots & 2^{n-2}-1 \\ 0 & 0 & 0 & 1 & 3 & \cdots & 2^{n-3}-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3 \end{pmatrix}$$

Proof: For v^{i+1}, consider a vertex c_{i+1} inserted between c_1 and c_i. Assume that the vector v^i, containing the number of connected graphs for each possible visibility degree of c_i, is known. From the definition of the generating tree of connected graphs, and the relation $v^{i+1} = C_n v^i$ we can determine the entries of C_n as follows:

- **First row**. The number of connected graphs where c_{i+1} has degree 0 is equal to the number of all the connected graphs where c_i has degree t, $\quad t = 0, \ldots, n-2$ when adding an edge from c_{i+1} to c_1 and then adding edges from c_{i+1} to all the visible vertices from c_1 to $c_{i+1}^j \in F(c_{i+1})$, where $j \geq 1$, removing a subset of edges whose endpoints are between c_1 and c_{i+1}^j, maintaining connectivity. This number of graphs is $\sum_{r=0}^{t-1} 2^r = 2^{t-1} - 1$, so this gives a first row of powers of 2 minus 1 in the matrix.

- **Second row**. The number of connected graphs where c_{i+1} has degree 1 and c_i has degree 0 is equal to the number of connected graph connecting c_{i+1} with c_i, thus we get a one in the first column of the second row. As soon as c_i has degree at least 1, we have to add one edge from c_{i+1} to c_{i+1}^2, and then we add edges from c_{i+1} to all the visible vertices from c_{i+1}^2 to c_{i+1}^j in $F(c_{i+1})$, where $j \geq 2$, removing a subset of edges whose endpoints are between c_{i+1}^2 and c_{i+1}^j. This number of graphs is $\sum_{r=0}^{t-2} 2^r = 2^{t-2} - 1$. Thus, the rest of the row is made of powers of 2 minus 1.

- **Other rows**. The cases when $|F(c_{i+1})| > |F(c_i)| + 1$ are not possible, so we get a zero in these cases.
  The following rows are analogous, shifted by one column every time: in order for c_{i+1} to have degree t, one edge need to be added from c_{i+1} to c_{i+1}^{t+1}, and then we can add a subset of edges from c_{i+1} to all the visible vertices from c_{i+1}^{t+1} to c_{i+1}^j in $F(c_{i+1})$, where $j \geq t+1$, removing a subset of edges whose endpoints are between c_{i+1}^{t+1} and c_{i+1}^j. So, we get a power of 2 minus 1 for each entry. $\square$

Let v^i be the vector of connected graphs (that is, v^i is the first column of a power of C_n for $i < n$; the sum of the elements of v^i is the number of connected graphs) The first vectors are:

$$v^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^3 = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^4 = \begin{pmatrix} 16 \\ 6 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^5 = \begin{pmatrix} 105 \\ 41 \\ 9 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

If we sum up all the entries of each vector v^i, we can verify that the sequence of numbers is the number given by Sequence $A007297$ in the On-Line Encyclopedia of Integer Sequences [41] obtaining the number of connected graphs on i labelled nodes on a circle with straight-line edges that do not cross.

In the following theorem we give a formula to calculate the entries of the vector v^n of connected graphs. Each entry v_j^n counts the number of connected graphs on n points in convex position and the root vertex has degree $j-1$:

Theorem 23 *Let v^n be the vector that counts the number of connected graphs on n vertices, then the k-th entry of such vector, $\forall k = 1, \ldots, n-1$ is:*

$$v_k^n = \frac{k}{n-1} 2^{n-1} \sum_{j=0}^{n-1} \binom{n-1}{j} \left(-\frac{1}{2}\right)^j \sum_{\ell=0}^{n-k-1} \binom{n-2-j+\ell}{\ell} \binom{j+n-\ell-k-2}{n-\ell-k-1} 2^\ell$$

Proof: The matrix C_n of connected graphs has a corresponding Riordan Array. The second row of C_n gives the A-sequence $\{1, 3, 7, 15, 31, \dots\}$ with generating function $A(t) = \frac{1}{(1-2t)(1-t)}$ and its Z-sequence is given by the first row of C_n and is $\{3, 7, 15, 31, 63, \dots\}$ with generating function $Z(t) = \frac{3-2t}{(1-2t)(1-t)}$. We have $d_0 = 1$ and

$$h(t) = A(th(t)) = \frac{1}{(1-2th(t))(1-th(t))}$$

We substitute $th(t) = w$. Then we get

$$w = \frac{t}{(1-2w)(1-w)}$$

We define a function $\phi(w) = \frac{1}{(1-2w)(1-w)}$. Then $w = t\phi(w)$ and we can apply Lagrange inversion. But first, we calculate $d(t)$:

$$d(t) = \frac{d_0}{1-tZ(th(t))} = \frac{1}{1-t\left(\frac{3-2th(t)}{(1-2th(t))(1-th(t))}\right)} = \frac{(1-2th(t))(1-th(t))}{(1-2th(t))(1-th(t))-3t+2t^2h(t)}$$

We substitute $th(t) = w$ and obtain

$$d(t) = \frac{h(t)(1-2w)(1-w)}{h(t)(1-2w)(1-w)-3w+2w^2} = \dots = \frac{1}{(1-2w)(1-w)}$$

From this result and [38, Theorem 3.8] we see that $d(t) = h(t)$. Then, Theorem 3 states that $d_{n,k} = \frac{k+1}{n+1}[t^{n-k}]A(t)^{n+1}$. It follows from this theorem together with [38, Theorem 3.10], (and a shift of index) that the k-th entry of v_n is the coefficient of t^{n-k-1} in the Taylor expansion of $\frac{k}{n-1}\left(\frac{1}{(1-2t)(1-t)}\right)^{n-1}$. Firstly, we use the partial fraction decomposition, and divide the generating function $A(t)$ into two terms in order to apply the binomial theorem to them.

$$\begin{aligned}
A(t)^{n-1} = \left(\frac{1}{(1-2t)(1-t)}\right)^{n-1} &= \left(\frac{2}{1-2t} - \frac{1}{1-t}\right)^{n-1} = \\
&= \sum_{j=0}^{n-1}\binom{n-1}{j}\left(\frac{2}{1-2t}\right)^{n-1-j}\left(\frac{-1}{1-t}\right)^{j} = \\
&= 2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}2^{-j}(-1)^j\left(\frac{1}{1-2t}\right)^{n-1-j}\left(\frac{1}{1-t}\right)^{j}
\end{aligned}$$

Secondly, we apply the identity given by (3.4) so the previous equation is equal to

$$
\begin{aligned}
A(t)^{n-1} &= 2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}\left(-\frac{1}{2}\right)^j\sum_{\ell=0}^{\infty}\binom{n-j+\ell-2}{\ell}(2t)^\ell\sum_{m=0}^{\infty}\binom{j+m-1}{m}t^m = \\
&= 2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}\left(-\frac{1}{2}\right)^j\sum_{\ell=0}^{\infty}\binom{n-j+\ell-2}{\ell}2^\ell\sum_{m=0}^{\infty}\binom{j+m-1}{m}t^{\ell+m}
\end{aligned}
$$

It remains to determine the coefficient of t^{n-k-1}. We set $n-k-1=\ell+m$ and therefore $m=n-\ell-k-1$.

$$
A(t)^{n-1} = 2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}\left(-\frac{1}{2}\right)^j\sum_{\ell=0}^{\infty}\binom{n-j+\ell-2}{\ell}(2t)^\ell\binom{j+n-\ell-k-2}{n-\ell-k-1}t^{n-k-\ell-1}
$$

Note that in the summation we can put $\ell \le n-k-1$. We thus arrive at the claimed formula. So after some substitutions, we arrive at the desired formula:

$$
v_k^n = \frac{k}{n-1}2^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}\left(-\frac{1}{2}\right)^j\sum_{\ell=0}^{n-k-1}\binom{n-2-j+\ell}{\ell}\binom{j+n-\ell-k-2}{n-\ell-k-1}2^\ell
$$

□

5.2 Characteristic polynomial

Now, we calculate another new result regarding the production matrix C_n. This new result is a recursive formula for the characteritic polynomial $c_n(\lambda)$ of C_n given by the following theorem, which can be derived from Theorem 6 with $a_{-1}=1$ and $a_{i-1}=2^{i+1}-1$.

The sequence $\{c_n(\lambda)\}$ starts with

$n=1$	$3-\lambda$
$n=2$	$\lambda^2-6\lambda+2$
$n=3$	$-\lambda^3+9\lambda^2-13\lambda$
$n=4$	$\lambda^4-12\lambda^3+33\lambda^2-12\lambda$
$n=5$	$-\lambda^5+15\lambda^4-62\lambda^3+63\lambda^2-4\lambda$
$n=6$	$\lambda^6-18\lambda^5+100\lambda^4-180\lambda^3+66\lambda^2$

Corollary 12 *The characteristic polynomial $c_n(\lambda)$ of the matrix C_n satisfies the recurrence relation*

$$
c_n(\lambda) = (3-\lambda)c_{n-1}(\lambda) - \sum_{i=2}^{n}(-1)^i(2^{i+1}-1)c_{n-i}(\lambda)
$$

Once we have a recurrence relation for the characteristic polynomial, we proceed to compute the solution for the recurrence, obtaining the following result:

Theorem 24 *The solution of the recurrence relation*

$$c_n(\lambda) = (3-\lambda)c_{n-1}(\lambda) - \sum_{i=2}^{n}(-1)^i(2^{i+1}-1)c_{n-i}(\lambda)$$

with initial condition $c_0(\lambda) = 1$ *is*

$$c_n(\lambda) = \sum_{t=0}^{n}\left(\sum_{\ell=0}^{t}\binom{t}{\ell}(-1)^t 2^{t-\ell}3^{n+2\ell-3t-2}\left[2\binom{\ell}{n+2\ell-3t-2}+9\binom{\ell+1}{n+2\ell-3t}\right]\right)\lambda^t$$

Proof: Consider the infinite matrix

$$M = \begin{pmatrix} 3-\lambda & -7 & 15 & -31 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

and let w_0 be the vector $(1, 0, \ldots)^\top$, and let w_i be the vector whose first i entries are the first i characteristic polynomials $c_i(\lambda)$, and the remaining ones are zero. That is, $w_i = (c_i(\lambda), c_{i-1}(\lambda), c_{i-2}(\lambda), \ldots, c_1(\lambda), 0, \ldots)^\top$. Then $M \cdot w_i = w_{i+1}$. It follows that w_n is the first column of M^n. We now can use the Riordan Array approach. The Z-sequence is $\{3-\lambda, -7, 15, \ldots\}$ with generating function $Z(t) = (3-\lambda) + \sum_{k=1}^{\infty}(-1)^k(2^{k+2}-1)t^k = (3-\lambda) + 4\sum_{k=1}^{\infty}(-1)^k(2t)^k - \sum_{k=1}^{\infty}(-1)^k t^k = -\lambda + \frac{4}{1+2t} - \frac{1}{1+t}$. The A-sequence is $\{1, 0, \ldots\}$ with generating function $A(t) = 1$. It follows that $h(t) = 1$ and

$$d(t) = \frac{1}{1 - t(-\lambda + \frac{4}{1+2t} - \frac{1}{1+t})} = \frac{1}{1+t(\lambda(1+3t+2t^2))} + \frac{3t}{1+t(\lambda(1+3t+2t^2))} + \\ + \frac{2t^2}{1+t(\lambda(1+3t+2t^2))} = d_1(t) + 3td_1(t) + 2t^2 d_1(t)$$

where

$$d_1(t) = \frac{1}{1+t(\lambda(1+3t+2t^2))}$$

Then,

$$d_{n,k} = [t^n]d_1(t)(th(t))^k + 3[t^{n-1}]d_1(t)(th(t))^k + 2[t^{n-2}]d_1(t)(th(t))^k \tag{5.1}$$

If we set $z = \lambda(1+3t+2t^2)$, then we know that

$$\frac{1}{1+z} = \sum_{k=0}^{\infty}(-1)^k z^k$$

Therefore,

$$d_1(t) = \sum_{k=0}^{\infty} t^k (-1)^k (\lambda(1+3t+2t^2))^k$$

We apply the binomial theorem to $(1+\lambda(1+3t+2t^2))^k$. We then only have to take the coefficient of t^n in $d_1(t)$ for the first term of (5.1), the coefficient of t^{n-1} in $d_1(t)$ for the second term of (5.1) and the coefficient of t^{n-2} for the third term of (5.1). This, after some short calculations, gives the formula:

$$\begin{aligned} c_n(\lambda) = & \sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^k \binom{k}{\ell}\binom{\ell}{n+2\ell-3k} 2^{k-\ell}3^{n+2\ell-3k}\lambda^k \\ & + \sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^k \binom{k}{\ell}\binom{\ell}{n+2\ell-3k-1} 2^{k-\ell}3^{n+2\ell-3k-1}\lambda^k \\ & + \sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^k \binom{k}{\ell}\binom{\ell}{n+2\ell-3k-2} 2^{k-\ell+1}3^{n+2\ell-3k-2}\lambda^k \end{aligned}$$

We finally rewrite $c_n(\lambda) = \sum_{t=0}^{n} c_t\lambda^t$, with c_t the coefficient of λ^t. Since $\lambda^t = \lambda^k$ we set $k = t$ and get after some calculations:

$$c_n(\lambda) = \sum_{t=0}^{n}\left(\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^t 2^{k-\ell}3^{n+2\ell-3t-2}\left[2\binom{\ell}{n+2\ell-3t-2} + 9\binom{\ell+1}{n+2\ell-3t}\right]\right)\lambda^t$$

□

5.3 Eigenvectors of C_n

In the following section we give a formula for the entries of an eigenvector x associated to the matrix C_n and the eigenvalue λ.

Corollary 13 *Let $x = (x_{n-1}, x_{n-2}, \ldots, x_0)$ be an eigenvector associated to an eigenvalue λ of the production matrix C_n. Then the entries of the vector x are of the form:*

$$x_i = (-1)^i c_i(\lambda)x_0$$

where $c_i(\lambda)$ is the characteristic polynomial of C_i.

We can derived this corollary from Theorem 6 in Section 2 with $a_{-1} = 1$.

5.4 Other properties

In this section, we present a conjecture related to the characteristic polynomial. This conjecture gives a relation between some well-known sequence of numbers and the entries of the vector that counts the number of connected graphs. It has been obtained thanks to the Krylov matrix introduced in Section 3.1.5. The Krylov matrix X_n associated to the matrix C_n of connected graphs and the vector $v^1 = (1, 0, 0, 0, \ldots, 0)^\top$ is featured below:

$$X_n = \begin{pmatrix} 1 & 3 & 16 & 105 & 768 & \cdots & v_1^n \\ 0 & 1 & 6 & 41 & 306 & \cdots & v_2^n \\ 0 & 0 & 1 & 9 & 75 & \cdots & v_3^n \\ 0 & 0 & 0 & 1 & 12 & \cdots & v_4^n \\ 0 & 0 & 0 & 0 & 1 & \cdots & v_5^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where v_j^i is the number of connected graphs on i nodes and root vertex of degree $j-1$.

Conjecture 1 *Let X_n be the Krylov matrix associated to the production matrix C_n, then the inverse of X_n has the following form:*

$$X_n^{-1} = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & \cdots & x_{1,n} \\ 0 & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & \cdots & x_{2,n} \\ 0 & 0 & x_{3,3} & x_{3,4} & x_{3,5} & \cdots & x_{3,n} \\ 0 & 0 & 0 & x_{4,4} & x_{4,5} & \cdots & x_{4,n} \\ 0 & 0 & 0 & 0 & x_{5,5} & \cdots & x_{5,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & x_{n,n} \end{pmatrix}$$

with entries $x_{i,j} = (-1)^{i+j} \sum_{k=0}^{i} \binom{i}{3i-j-k} \binom{i}{k} 2^{i-k}$

5.5 Relation with geometric graphs

Finally, in order to establish a relation between the class of connected geometric graphs and the class of all geometric graphs, we start by creating a matrix whose entries contain the number c_i of connected graphs on i vertices. First of all, we start by studying the relation between geometric graphs and connected graphs.

In this section, we take into account another definition of the root vertex degree. Let $\{r_1, r_2, \ldots, r_n\}$ be a set of vertices, ordered counter-clockwise in convex position. Then, the degree of the root vertex r_n is defined as the number of isolated vertices visible from a vertex r_{n+1} inserted in convex position between r_1 and r_n in convex position. For instance, if we have a look at Figure 5.3, we observe that the isolated vertices visible from a vertex inserted between r_1 and r_{12} are r_8 and r_{12}, so the degree of the root vertex r_{12} is 2.

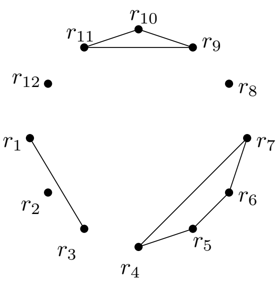

Figure 5.3: Geometric graph with $n = 12$ vertices.

In general, for a given geometric graph, we generate its children by leaving it isolated or by connecting the new root vertex with a subset of the isolated vertices visible from it obtaining a geometric graph with this subset of vertices. We can see an example in the following figure:

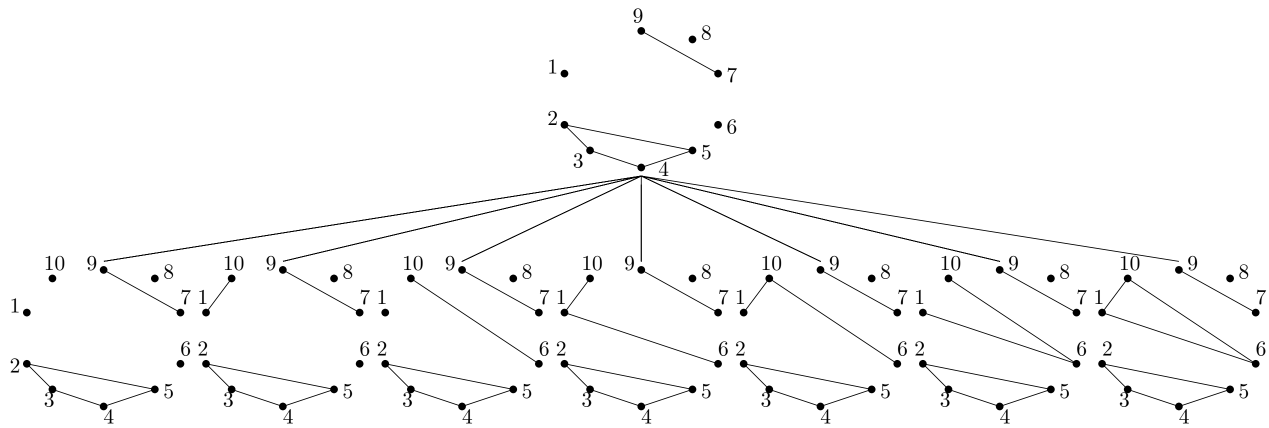

Figure 5.4: Children of a given geometric graph in the tree of geometric trees.

As we have said in the introduction, the vector v^{i+1} that counts the number of geometric graphs on $i+1$ vertices is obtained from the multiplication of the production matrix with the vector v^i that counts the number of geometric graphs on i vertices. In the case of geometric graphs, we have the following relation:

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i+1} = R_n \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i}$$

where R_n is the production matrix that counts geometric graphs whose entries are the number of connected graphs on a certain number of vertices, and d_j^i is the number of graphs on i vertices and root vertex degree of $j-1$.

Let $I(r_{i+1}) = \{r_1 = r_{i+1}^1, r_{i+1}^2, \ldots, r_{i+1}^m = r_i\},\ m < i+1$ be the ordered sequence of the vertices visible from a point r_{i+1} inserted between r_1 and r_i. Then we obtain the next result:

Theorem 25 *The following $n \times n$ matrix R_n is a production matrix for non-crossing geometric graphs of point sets in convex position.*

$$R_n = \begin{pmatrix} 0 & c_2 & c_2+c_3 & c_2+2c_3+c_4 & c_2+3c_3+3c_4+c_5 & \cdots & \sum_{i=2}^{n}\binom{n-2}{i-2}c_i \\ 1 & 0 & c_2 & c_2+c_3 & c_2+2c_3+c_4 & \cdots & \sum_{i=2}^{n}\binom{n-3}{i-2}c_i \\ 0 & 1 & 0 & c_2 & c_2+c_3 & \cdots & \sum_{i=2}^{n}\binom{n-4}{i-2}c_i \\ 0 & 0 & 1 & 0 & c_2 & \cdots & \sum_{i=2}^{n}\binom{n-5}{i-2}c_i \\ 0 & 0 & 0 & 1 & 0 & \cdots & \sum_{i=2}^{n}\binom{n-6}{i-2}c_i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where c_i, $i = 2, \ldots, n$ is the number of connected graphs on i vertices.

Proof: For v^{i+1}, consider a vertex r_{i+1} inserted between r_1 and r_i. Assume that the vector v^i containing the number of connected graphs for each possible visibility degree of r_n, is known. From the definition of the generating tree and the relation $v^{i+1} = R_n v^i$ we can determine the entries of R_n as follows:

- **Main diagonal**. Observe first that, when adding the point r_{i+1} to a geometric graph with the degree of the root vertex r_i equal to k, we cannot obtain a geometric graph with root vertex of degree k, so this gives a diagonal of 0s in R_n.

- **First subdiagonal**. We may obtain one geometric graph with $I(r_{i+1}) = k+1$ from a geometric graph with $I(r_i) = k$ by leaving the new vertex isolated, so this gives the first subdiagonal of 1s.

- **Other diagonals**. The cases where $|I(r_{i+1})| > |I(r_i)| + 1$ are not possible, so we get a 0 in these cases.
  In general, for any geometric graph on i points, we can decide whether to keep r_{i+1} isolated or to connect it to a subset of vertices of $I(r_{i+1})$ as follows: if we want $|I(r_{i+1})| = s$ with $s < m+1$, we have to connect r_{i+1} with r_{i+1}^{s+1}, and then we have $\binom{m-s-1}{1}c_3$ ways of connecting r_{i+1} and r_{i+1}^{s+1} with one of the $m-s-1$ vertices visible from r_{i+1}, also we have $\binom{m-s-1}{2}c_4$ ways of connecting r_{i+1} and r_{i+1}^{s+1} with a pair of the $m-s-1$ vertices visible from r_{i+1}, and so on. This yields the claimed matrix.
  □

If we compare the matrix R_n, substituting each entry c_i by the number of connected graphs on i vertices given by Theorem 23, with the production matrix G_n obtained in the previous chapter in Theorem 19, we realize that they are different. So we have obtained a new matrix that enumerates the number of geometric graphs.

$$R_n = \begin{pmatrix} 0 & 1 & 5 & 32 & 238 & \cdots \\ 1 & 0 & 1 & 5 & 32 & \cdots \\ 0 & 1 & 0 & 1 & 5 & \cdots \\ 0 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let v^i be the vector of geometric graphs (that is, v^i is the first column of a power of R_n for $i < n$; the sum of the elements of v^i is the number of geometric graphs). The first vectors are:

$$v^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^3 = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^4 = \begin{pmatrix} 34 \\ 10 \\ 3 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \qquad v^5 = \begin{pmatrix} 263 \\ 69 \\ 15 \\ 4 \\ 0 \\ 1 \\ \vdots \end{pmatrix}$$

If we sum up all the entries of each vector v^i, we can verify that the sequence of numbers is the same as the previous result about the number of geometric graphs. Nevertheless, the entries of the vectors are different.

However, changing the class of connected graphs and non-connected graphs, we realize that this new matrix also works for other graph classes. For instance, if we substitute in R_n each entry c_i by the number of spanning trees on i vertices, we obtain the following matrix:

Theorem 26 *The following $n \times n$ matrix R_n is a production matrix for non-crossing forests of point sets in convex position*

$$R_n = \begin{pmatrix} 0 & 1 & 4 & 19 & 101 & \cdots \\ 1 & 0 & 1 & 4 & 19 & \cdots \\ 0 & 1 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This matrix R_n calculates the number of forests of rooted trees with n nodes on a circle without crossing edges given by Sequence $A054727$ in the On-Line Encyclopedia of Integer Sequences [41]. Furthermore, if we substitute in R_n the entries c_i by the number of spanning paths with i vertices, we obtain the following matrix:

Theorem 27 *The following $n \times n$ matrix R_n is a production matrix for non-crossing forests of paths of point sets in convex position*

$$R_n = \begin{pmatrix} 0 & 1 & 4 & 15 & 54 & \cdots \\ 1 & 0 & 1 & 4 & 15 & \cdots \\ 0 & 1 & 0 & 1 & 4 & \cdots \\ 0 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This new matrix R_n calculates the number of labelled forests on n nodes each component of which is a path, given by Sequence $A011800$ in the On-Line Encyclopedia of

Integer Sequences [41].

As further work, we propose to find formulas that allow to calculate the entries of a vector v^n that counts the number of graphs given by the matrix R_n given by Theorems 26 and 27, as well as other properties of these matrices such as the characteristic polynomial, an eigenvector or the eigenvalues associated to each eigenvector.

Chapter 6

Spanning Trees

A tree is a graph in which any two vertices are connected by exactly one path. In other words, any acyclic connected graph is a tree. In our case, all the vertices are in convex position and the tree is without edge crossings.

6.1 Production matrix

In previous papers [29], a matrix S_n for the number of spanning trees with given root vertex degree has been given. Also see Chapter 3 on quadrangulations. The degree of the root vertex is defined as the number of incident edges to it. Hence, the matrix S_n is defined as follows:

$$S_n = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & \cdots & n+1 \\ 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 0 & 1 & 2 & 3 & 4 & \cdots & n-1 \\ 0 & 0 & 1 & 2 & 3 & \cdots & n-2 \\ 0 & 0 & 0 & 1 & 2 & \cdots & n-3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

Also in [29], they give a formula for the number of spanning trees with point sets in convex position and the degree of the root vertex defined as follows: the number of incident edges to it. This well-known formula is:

$$v_k^n = \frac{k}{n-1}\binom{3n-k-4}{n-k-1} \tag{6.1}$$

where n is the number of vertices in which the root vertex has degree k. And we have that $v^{i+1} = S_n v^i$ holds for $i < n$.

However, as we have said previously, there are other ways to define the degree of a root vertex. So, in this chapter, the degree of the root vertex is defined in a different way, based on visibility. Let $\{s_1, s_2, \ldots, s_n\}$ be a set of points in convex position. Then the degree of the root vertex s_n is the number of vertices visible from a vertex s_{n+1} inserted in convex position between s_1 and s_n, minus 2. Two vertices are visible if the line segment connecting them does not intersect the interior of any edge of the graph. For this class of graphs, we obtain a production matrix S_n that fulfills the following:

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i+1} = S_n \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i}$$

where d_j^i is the number of spanning trees on i nodes with root vertex of degree $j-1$.

For instance, if we have a look at the spanning trees of Figure 6.1, we observe that in the right graph, the degree of the root vertex s_{12} with both definitions coincides, and it is 3. However, in the left graph, with the degree definition of [29], the root vertex s_{12} has visibility degree 3, and with this new definition s_{12} has visibility degree 1.

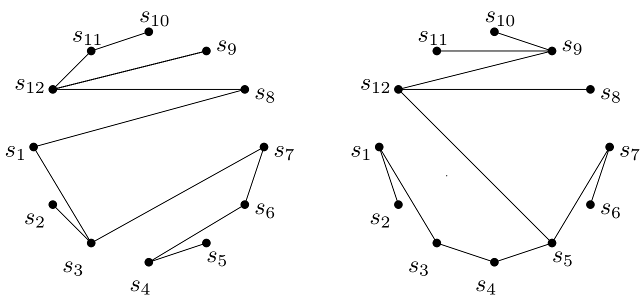

Figure 6.1: Spanning tree with $n = 12$ vertices.

Let $F(s_{i+1}) = \{s_1 = s_{i+1}^1, s_{i+1}^2, \ldots, s_{i+1}^k = s_i\}, \quad k < i+1$ be the ordered sequence of the vertices visible from s_{i+1}. Figure 6.2 shows how we can obtain the children of a given spanning tree by connecting the new root vertex with a subset of its visible vertices and delete the edge $s_j s_{j+1}$ is we have add the edges $s_{i+1}s_j$ and $s_{i+1}s_{j+1}$.

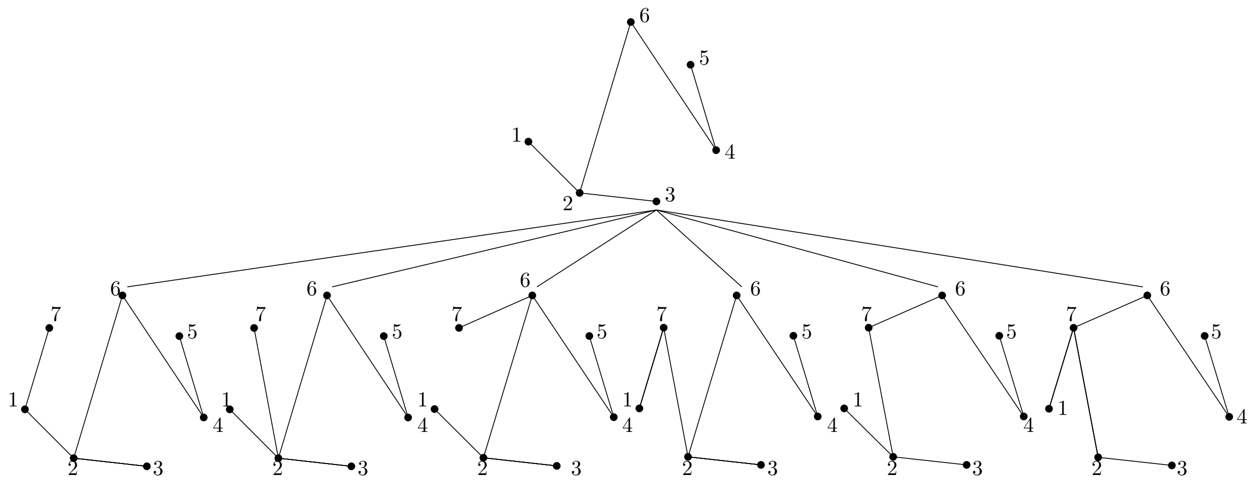

Figure 6.2: Children of a given spanning tree in the tree of spannig trees.

Then the matrix S_n is defined in the following theorem:

Theorem 28 *The following $n \times n$ matrix S_n is a production matrix for spanning trees of point sets in convex position*

$$S_n = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & \cdots & n+1 \\ 1 & 2 & 3 & 4 & 5 & \cdots & n \\ 0 & 1 & 2 & 3 & 4 & \cdots & n-1 \\ 0 & 0 & 1 & 2 & 3 & \cdots & n-2 \\ 0 & 0 & 0 & 1 & 2 & \cdots & n-3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

Proof: For v^{i+1}, consider a vertex s_{i+1} inserted between s_1 and s_i. Assume that the vector v^i, containing the number of spanning trees for each possible degree of s_i, is known. From the definition of the generating tree and the relation $v^{i+1} = S_n v^i$ we can determine the entries of S_n as follows:

- **First row**. The number of spanning trees where s_{i+1} has degree 0 is equal to all the spanning trees where s_i has degree t, $\quad t = 0, \ldots, n-2$, when adding an edge from s_{i+1} to s_1; and then we can connect s_{i+1} to a subset of visible vertices from s_{i+1}^j to s_{i+1}^k where $s_{i+1}^j, s_{i+1}^k \in F(s_{i+1})$ for each $j \leq k$, and removing the edges whose endpoints are between s_{i+1}^j and s_{i+1}^k in $F(s_{i+1})$. This gives a first row with entries $t+2$ in the matrix.

- **Second row**. The number of spanning trees where s_{i+1} has degree 1, obtained from the spanning trees where s_i has degree 0 is equal to the spanning tree connecting s_{i+1} with s_i. Thus, we get a one in the first column of the second row. As soon as s_i has degree at least 1, we have to add one edge from s_{i+1} to s_{i+1}^2; and then we can connect s_{i+1} to a subset of its visible vertices from s_{i+1}^j to s_{i+1}^k where $s_{i+1}^j, s_{i+1}^k \in F(s_{i+1})$ for each $2 < j \leq k$, and removing the edges whose endpoints are between s_{i+1}^j and s_{i+1}^k in $F(s_{i+1})$. Thus, the rest of the row is made of the natural numbers up to n.

- **Other rows**. The cases where $|F(s_{i+1})| > |F(s_i)| + 1$ are not possible, so we get a zero in these cases.
  The following rows are analogous, shifted by one column every time: in order for s_{i+1} to have degree t, one edge needs to be added from s_{i+1} to s_{i+1}^{t+1}; and then we can connect s_{i+1} to a subset of its visible vertices from s_{i+1}^j to s_{i+1}^k where $s_{i+1}^j, s_{i+1}^k \in F(s_{i+1})$ for each $t < j \leq k$, and removing the edges whose endpoints are between s_{i+1}^j and s_{i+1}^k in $F(s_{i+1})$. $\square$

If we compare this matrix with the one we knew before [29], we realize that they are the same. Let v^i be the vector of spanning trees (that is, v^i is the first column of a power of S_n for $i < n$; the sum of the elements of v^i is the number of spanning trees). The first vectors are:

$$v^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^3 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^4 = \begin{pmatrix} 7 \\ 4 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^5 = \begin{pmatrix} 30 \\ 18 \\ 6 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

If we sum up all the entries of each vector v^i, we can verify that the sequence of numbers is the same as the result in [29] for spanning trees and the same for quadrangulations obtained in this work and given by Sequence $A001764$ in the On-Line Encyclopedia of Integer Sequences [41].

Clearly, also the other results for the production matrices such as the formula for the number of spanning trees and the characteristic polynomial are equal to the ones we already knew [29].

Chapter 7

Non-crossing partitions

Given a set of n elements, which we will assume to be $[n] = \{1, 2, \ldots, n\}$, a partition of $[n]$ is a family of nonempty, pairwise disjoint sets $B_1, B_2, \ldots, B_k$, called blocks, whose union is the n-element set. A partition of $[n]$ is non-crossing if whenever four elements, $1 \leq a < b < c < d \leq n$, are such that if a, c are in the same block and b, d are in the same block, then the two blocks coincide. In our case, we treat non-crossing partitions with the vertices in convex position. Figure 7.1 shows an example of a non-crossing partition.

7.1 Production matrix

There exist previous research papers where the production matrix of non-crossing partitions has been given like in [29], where we can find a relation between the number of triangulations and the number of matchings. The matrix given in that work is the following.

$$T_n = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{7.1}$$

However, in this chapter we give a new production matrix for this type of graphs. Let $\{b_1, b_2, \ldots, b_n\}$ be a set of points in convex position. Whereas in the paper of Huemer et al. [29] the degree of the root vertex b_n is defined as the number of blocks visible from the root vertex, in this thesis, we define the degree of the root vertex b_n as the number of *isolated* vertices visible from a vertex b_{n+1} inserted between b_1 and b_n in convex position.

For a given non-crossing partition, we generate its children by leaving it isolated or by connecting the new root vertex b_{n+1} with a subset of its visible vertices obtaining a partition between them. We can find an example in the following figure:

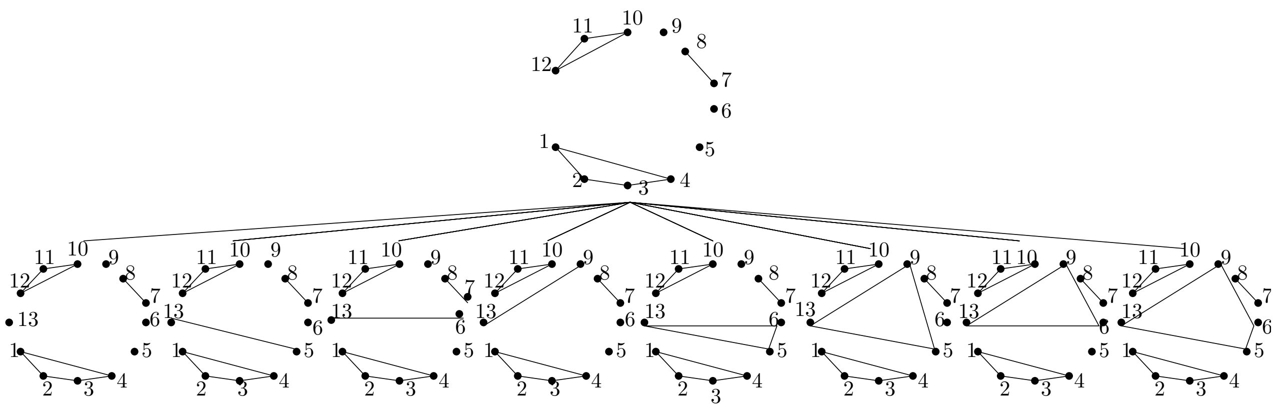

Figure 7.1: Children of a given non-crossing partition in the tree of non-crossing partitions.

So, to find the production matrix B_n of non-crossing partitions we have defined the following:

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i+1} = B_n \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i}$$

where d_j^i is the number of non-crossing partitions on i vertices with root vertex of degree $j-1$, and B_n is the production matrix for non-crossing partitions.

Let $I(b_{i+1}) = \{b_1 = b_{i+1}^1, b_{i+1}^2, \ldots, b_{i+1}^k = b_i\}, \quad k < i+1$ be the ordered sequence of the isolated vertices visible from b_{i+1}. Then we obtain the next result:

Theorem 29 *The following $n \times n$ matrix B_n is a production matrix for non-crossing partitions of point sets in convex position.*

$$B_n = \begin{pmatrix} 0 & 1 & 2 & 4 & 8 & \cdots & 2^{n-2} \\ 1 & 0 & 1 & 2 & 4 & \cdots & 2^{n-3} \\ 0 & 1 & 0 & 1 & 2 & \cdots & 2^{n-4} \\ 0 & 0 & 1 & 0 & 1 & \cdots & 2^{n-5} \\ 0 & 0 & 0 & 1 & 0 & \cdots & 2^{n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Proof: For v^{i+1}, consider a vertex b_{i+1} inserted between b_1 and b_i. Assume that the vector v^i, containing the number of non-crossing partitions for each possible degree of b_i, is known. From the definition of the generating tree and the relation $v^{i+1} = B_n v^i$ we can determine the entries of B_n as follows:

- **First row**. The number of non-crossing partitions where b_{i+1} has degree 0 is equal to the number of non-crossing partitions where b_i has degree $k, \quad t = 0, \ldots, n-2$, adding an edge from b_{i+1} to b_1; and then we can add a subset of the isolated visible vertices from b_{i+1} to the block containing b_{i+1}. This gives a first row with powers of 2 in the matrix. And this is because we have $\binom{k-1}{1}$ ways of producing a block of size 1 with the vertex b_{i+1} and the nodes of $I(b_{i+1})$, $\binom{k-1}{2}$ blocks of size 2 with b_{i+1} and the vertices of $I(b_{i+1})$, and so on.

- **First subdiagonal**. If $|I(b_{i+1})| = |I(b_i)| + 1$, we obtain one non-crossing partition by connecting the new root b_{i+1} with b^k_{i+1}, so this gives a first subdiagonal of 1s in the matrix.

- **Other rows**. The cases where $|I(b_{i+1})| > |I(b_i)| + 1$ are not possible, so we get a zero in these cases.
  It is also impossible to obtain a non-crossing partition with root vertex of degree r from a non-crossing partition with root degree r.
  The following rows are analogous, shifted by one column every time: in order for b_{i+1} to have degree t, one edge need to be added from b_{i+1} to b^{t+1}_{i+1}, and then we connect b_{i+1} to a subset of its isolated visible vertices obtaining convex hulls. $\square$

Let v^i be the vector of non-crossing partitions (that is, v^i is the first column of a power of B_n for $i < n$; the sum of the elements of v^i is the number of non-crossing partitions). The first vectors are:

$$v^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^3 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \qquad v^4 = \begin{pmatrix} 6 \\ 4 \\ 3 \\ 0 \\ 1 \\ \vdots \end{pmatrix}$$

If we sum up all the entries of each vector v^i, we can verify that the sequence of numbers is the Catalan numbers, the same sequence given by the matrix (7.1) that counts the number of triangulations with the vertices in convex position as well as the number of perfect matchings. We can find this sequence of numbers in the On-Line Encyclopedia of Integer Sequence [41] with Sequence $A000108$. Thanks to this new matrix, we are able to count the Catalan numbers in a different way. So we present a new formula for these vectors v^i, whose entries are the different from the well-known Ballot numbers.

The following theorem gives the number of non-crossing partitions with n points in convex position and root vertex of degree $j-1$. The degree is based on visibility: the degree of b_n is the number of isolated vertices visible from a new node b_{n+1} inserted between b_1 and b_n.

Theorem 30 *Let v^n be the vector that counts the number of non-crossing partitions with n vertices, then the k-th entry of such vector, $\forall k = 1, \dots, n-1$ is:*

$$v^n_k = \frac{k}{n-1} \sum_{j=0}^{n-1} \binom{n-1}{j} \binom{n-j-k-2}{n-2j-k-1} 2^{n-2j-k-1}$$

Proof: The matrix B_n of non-crossing partitions has a corresponding Riordan Array. The second row of B_n gives the A-sequence $\{1, 0, 1, 2, 4, \dots\}$ with generating function

$A(t) = \frac{(t-1)^2}{1-2t}$ and its Z-sequence is given by the first row of B_n and is $\{0, 1, 2, 4, 8, \ldots\}$ with generating function $Z(t) = \frac{t}{1-2t}$. We have $d_0 = 1$ and

$$h(t) = A(th(t)) = \frac{(th(t)-1)^2}{1-2th(t)}$$

We substitute $th(t) = w$. Then we get

$$w = t\frac{(w-1)^2}{1-2w}$$

We define a function $\phi(w) = \frac{(w-1)^2}{1-2w}$. Then $w = t\phi(w)$ and we can apply Lagrange inversion. But first, we calculate $d(t)$:

$$d(t) = \frac{d_0}{1-tZ(th(t))} = \frac{1}{1-t(\frac{th(t)}{1-2th(t)})} = \frac{1-2th(t)}{1-2th(t)-t^2h(t)}$$

We substitute $th(t) = w$ and obtain

$$d(t) = \frac{h(t)(1-2w)}{h(t)-2wh(t)-w^2} = \cdots = \frac{(w-1)^2}{1-2w}$$

From the results in [38, Theorem 3.8] we see that $d(t) = h(t)$. Then [38, Theorem 3.4] states that $d_{n,k} = \frac{k+1}{n+1}[t^{n-k}]A(t)^{n+1}$. It follows from this theorem together with Theorem 3.10, (and a shift of index) that the k-th entry of v^n is the coefficient of t^{n-k-1} in the Taylor expansion of $\frac{k}{n-1}\left(\frac{(t-1)^2}{1-2t}\right)^{n-1}$. Using the well-known identity (3.4) we expand

$$\begin{aligned}
A(t)^{n-1} &= \left(\frac{(t-1)^2}{1-2t}\right)^{n-1} = \left(\frac{t^2}{1-2t}-1\right)^{n-1} = \sum_{j=0}^{n-1}\binom{n-1}{j}\left(\frac{t^2}{1-2t}\right)^j \\
&= \sum_{j=0}^{n-1}\binom{n-1}{j}t^{2j}\left(\frac{1}{1-2t}\right)^j = \sum_{j=0}^{n-1}\binom{n-1}{j}t^{2j}\sum_{\ell=0}^{\infty}\binom{j+\ell-1}{\ell}(2t)^\ell \\
&= \sum_{j=0}^{n-1}\sum_{\ell=0}^{\infty}\binom{n-1}{j}\binom{j+\ell-1}{\ell}2^\ell t^{\ell+2j}
\end{aligned}$$

It remains to determine the coefficient of $t^{\ell+2j}$. We set $n-k-1 = \ell+2j$ and therefore $\ell = n-k-1-2j$. We thus arrive at the claimed formula

$$v_k^n = \frac{k}{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}\binom{n-j-k-2}{n-2j-k-1}2^{n-2j-k-1}$$

□

7.2 Characteristic polynomial

The following section is devoted to show formulas for the characteristic polynomial of the production matrix B_n. The first result is a recurrence relation for such polynomial, which is given by Theorem 6 with $a_{-1} = 1$ and $a_{i-1} = 2^{i-2}$.

The sequence $\{b_n(\lambda)\}$ starts with

$$\begin{array}{r|c} n=1 & -\lambda \\ n=2 & \lambda^2-1 \\ n=3 & -\lambda^3+2\lambda+2 \\ n=4 & \lambda^4-3\lambda^2-4\lambda-3 \\ n=5 & -\lambda^5+4\lambda^3+6\lambda^2+5\lambda+4 \\ n=6 & \lambda^6-5\lambda^4-8\lambda^3-6\lambda^2-4\lambda-5 \end{array}$$

Corollary 14 *The characteristic polynomial $b_n(\lambda)$ of the matrix B_n satisfies the recurrence relation*

$$b_n(\lambda) = -\lambda b_{n-1}(\lambda) - \frac{1}{4}\sum_{i=2}^{n}(-2)^i b_{n-i}(\lambda)$$

Theorem 31 *The solution of the recurrence relation*

$$b_n(\lambda) = -\lambda b_{n-1}(\lambda) - \frac{1}{4}\sum_{i=2}^{n}(-2)^i b_{n-i}(\lambda)$$

with initial condition $b_0(\lambda) = 1$ is

$$b_n(\lambda) = \sum_{t=0}^{n}\sum_{k=0}^{n}\sum_{\ell=0}^{k}\binom{k}{t}\binom{t}{\ell}(-1)^k 2^{2k-n+t-2\ell}\left[4\binom{k-t}{2k-n-\ell+1}+\binom{k-t}{2k-n-\ell}\right]\lambda^t$$

Proof: Consider the infinite matrix

$$M = \begin{pmatrix} -\lambda & -1 & 2 & -4 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

and let w_0 be the vector $(1,0,\ldots)^\top$, and let w_i be the vector whose first i entries are the first i characteristic polynomials $b_i(\lambda)$, and the remaining ones are zero. That is, $w_i = (b_i(\lambda), b_{i-1}(\lambda), b_{i-2}(\lambda), \ldots, b_i(\lambda), 0, \ldots)$. Then $M \cdot w_i = w_{i+1}$. It follows that w_n is the first column of M^n. We now can use the Riordan Array approach. The Z-sequence is $\{-\lambda, -1, 2, -4, \ldots\}$ with generating function $Z(t) = -\lambda + \sum_{k=1}^{\infty}(-1)^k 2^{k-1}t^k = -\lambda + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k(2t)^k = -\lambda - \frac{t}{1+2t}$. The A-sequence is $\{1, 0, \ldots\}$ with generating function $A(t) = 1$. It follows that $h(t) = 1$ and

$$d(t) = \frac{1}{1-t(-\lambda-\frac{t}{1+2t})} = \frac{1}{1+t((2+\lambda)+(2\lambda+1)t)} + \frac{2t}{1+t((2+\lambda)+(2\lambda+1)t)}$$

Then,

$$d_{n,k} = [t^n]d_1(t)(th(t))^k + 2[t^{n-1}]d_1(t)(th(t))^k \tag{7.2}$$

where

$$d_1(t) = \frac{1}{1 + t((2+\lambda) + (2\lambda+1)t)}$$

If we set $z = (2+\lambda) + (2\lambda+1)t$, then we know that

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k$$

Therefore

$$d_1(t) = \sum_{k=0}^{\infty} t^k (-1)^k ((2+\lambda) + (2\lambda+1)t)^k$$

We apply the binomial theorem to $((2+\lambda)+(2\lambda+1)t)^k$. We then only have to take the coefficient of t^n in $d_1(t)$ for the left term of (7.2) and the coefficient of t^{n-1} in $d_1(t)$ for the right term of (7.2). This, after some short calculations, gives the formula

$$\begin{aligned} b_n(\lambda) = &\sum_{k=0}^{n} \sum_{\ell=0}^{2k-n} \sum_{i=0}^{n-k} (-1)^k \binom{k}{n-k} \binom{2k-n}{\ell} \binom{n-k}{i} 2^{2k-n-\ell+i} \lambda^{\ell+i} + \\ &+ 2 \sum_{k=0}^{n} \sum_{\ell=0}^{2k-n+1} \sum_{i=0}^{n-k-1} (-1)^k \binom{k}{n-k-1} \binom{2k-n+1}{\ell} \binom{n-k-1}{i} 2^{2k-n-\ell+i+1} \lambda^{\ell+i} \end{aligned}$$

Note that in the summation we can put $\ell \le k$

$$\begin{aligned} b_n(\lambda) = &\sum_{k=0}^{n} \sum_{\ell=0}^{k} \sum_{i=0}^{n-k} (-1)^k \binom{k}{n-k} \binom{2k-n}{\ell} \binom{n-k}{i} 2^{2k-n-\ell+i} \lambda^{\ell+i} + \\ &+ 2 \sum_{k=0}^{n} \sum_{\ell=0}^{k} \sum_{i=0}^{n-k-1} (-1)^k \binom{k}{n-k-1} \binom{2k-n+1}{\ell} \binom{n-k-1}{i} 2^{2k-n-\ell+i+1} \lambda^{\ell+i} \end{aligned}$$

we finally rewrite $b_n(\lambda) = \sum_{t=0}^{n} c_t(\lambda)\lambda^t$, with $c_t(\lambda)$ the coefficient of λ^t. Since $\lambda^t = \lambda^{\ell+i}$ we set $i = t - \ell$ and get

$$\begin{aligned} b_n(\lambda) = &\sum_{t=0}^{n} \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^k \binom{k}{n-k} \binom{2k-n}{\ell} \binom{n-k}{t-\ell} 2^{2k-n+t-2\ell} \lambda^t + \\ &+ \sum_{t=0}^{n} \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^k \binom{k}{n-k-1} \binom{2k-n+1}{\ell} \binom{n-k-1}{t-\ell} 2^{2k-2\ell-n+t+2} \lambda^t \end{aligned}$$

After some calculations, and taking into account the subset of a subset identity $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ we finally get the formula

$$b_n(\lambda) = \sum_{t=0}^{n} \sum_{k=0}^{n} \sum_{\ell=0}^{k} \binom{k}{t} \binom{t}{\ell} (-1)^k 2^{2k-n+t-2\ell} \left[4\binom{k-t}{2k-n-\ell+1} + \binom{k-t}{2k-n-\ell} \right] \lambda^t$$

□

7.3 Eigenvectors of B_n

In this section, we give a formula for the entries of an eigenvector x associated to the production matrix B_n and the eigenvalue λ.

Corollary 15 *Let $x = (x_{n-1}, x_{n-2}, \ldots, x_0)$ be an eigenvector associated to an eigenvalue λ of the productionmatrix B_n. Then the entries of the vector x are of the form:*

$$x_i = (-1)^i b_i(\lambda) x_0$$

where $b_i(\lambda)$ is the characteristic polynomial of B_i.

7.4 Non-crossing partitions with k blocks

Now, if we want to know the number of non-crossing partitions on n vertices and a given number of blocks k, we take into account a new definition of the degree of the root vertex. Here we define the degree of the root vertex b_n as the number of elements in the block of the element b_n, minus 1. In addition, the result we have obtained is different from the one previously known because we find two matrices F_n and H_n that give the number of non-crossing partitions on $n+1$ vertices and $k+1$ blocks.

Figure 7.2 shows the children of a given non-crossing partition. The way of obtaining them is by leaving the new root vertex b_{n+1} isolated or "stealing" vertices from the block of the former root b_n.

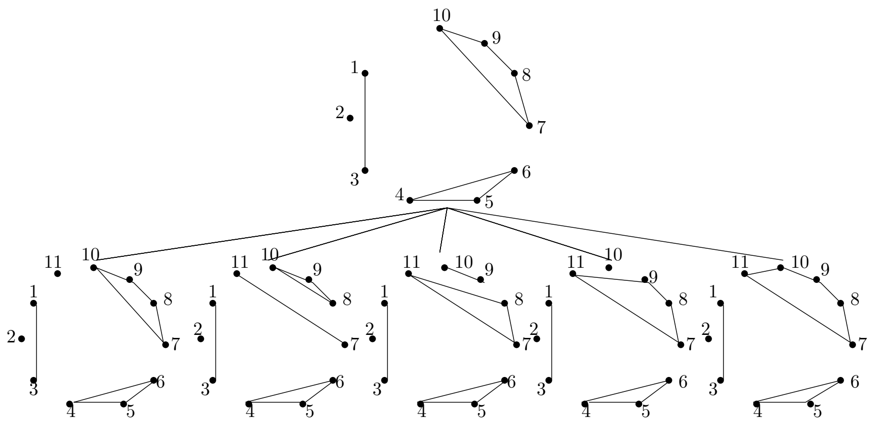

Figure 7.2: Children of a given non-crossing partition in the generating tree of non-crossing partitions. The right partition is produced by the matrix H, whereas the other ones are produced by the matrix F.

So, to find the matrices H_n and F_n of non-crossing partitions on k blocks we defined the following:

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i+1,k+1} = F_n \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i,k} + H_n \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \end{pmatrix}^{i,k+1}$$

where $d_j^{i,r}$ is the number of non-crossing partitions with a fixed number of blocks r, i vertices and with root vertex of degree $j-1$. The matrix F_n is the production matrix for non-crossing partitions in which the number of vertices n and the number of blocks k increases, the production matrix H_n is for non-crossing partitions in which only the number n of vertices increases.

Let $B(b_i) = \{b_1 = b_i^1, b_i^2, \dots, b_i^k = b_i\}$ be the ordered sequence of the vertices in the block containing b_n. So we obtain the next result:

Theorem 32 *The following $n \times n$ production matrices F_n and H_n are production matrices for non-crossing partitions of point sets in convex position with k and $k+1$ blocks, respectively.*

$$F_n = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \qquad H_n = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Proof: For the matrix F_n where we have to increase the number of nodes and the number of blocks, when adding a point b_{i+1} between b_1 and b_i, we can either keep the point as a singleton, or "steal" up to $|B(b_i)|-1$ points from $B(b_i)$, obtaining $|B(b_i)|-1$ partitions. By setting $v_j^{i,k}$ as the number of non-crossing partitions on i vertices, k blocks and root vertex degree of $|B(b_i)| = j$, the relation for $v^{i+1,k+1}$ is $v_j^{i+1,k+1} = \sum_{l \geq j} v_l^{i,k}$.

For the matrix H_n where we have to increase the number of nodes but not the number of blocks, when adding the new vertex b_{i+1} between the vertices b_1 and b_i, we can only obtain a new non-crossing partition with the same number of blocks if we add this new node to the same block as the former root b_i. This gives a first subdiagonal of 1's. □

As we have said in the introduction of the thesis, a production matrix A_n gives the relation $v^{i+1} = A_n v^i = A_n^{i+1-c} v^c$ when starting with a vector v^c for a constant number of vertices. This vector is usually the vector $(1,0,\dots,0)^\top$. However, these two new production matrices F_n and H_n give the relation $v^{i+1,j+1} = F_n v^{i,j} + H_n v^{i,j+1}$, so we need two vectors as base cases. These two vectors are $v^{i,i} = (1,0,\dots,0)^\top$ and $v^{i,1} = (0,\dots,0,1,0,\dots,0)^\top$, this last one, with the i-th entry a 1.

Let $v^{i,k}$ be the vector of non-crossing partitions given the number of blocks k (the sum of the elements of $v^{i,k}$ is the number of non-crossing partitions with a given number of blocks k). The first vectors are:

$$v^{1,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^{2,2} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^{2,1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^{3,2} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad v^{4,3} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

7.5 Bijections related to non-crossing partitions

If we sum up all the entries of each vector $v^{i,j}$, we can check that the sequence of numbers form the Narayana's triangle entries and it is given by Sequence $A001263$ in the On-Line Encyclopedia of Integer Sequences [41] that counts the ordered trees with n edges (i.e. $n+1$ nodes) and k leaves and the number of Dyck n-paths with exactly k peaks (i.e. a path in the first quadrant that begins at the origin, ends at $(2n, 0)$, and consists of steps $(1, 1)$ (North-East), called rises, and $(1, -1)$ (South-East), called falls). In a Dyck path a peak is an occurrence of rise-fall.

The bijection between this two problems and the problem of counting non-crossing partitions are defined as follow. Firstly, in [13], we can see the bijection between Dick paths and non-crossing partitions:

For a given non-crossing partition on $[n]$, traverse $[n]$ from 1 to n; to each i that is not the largest entry in its block there corresponds a rise step and to each i that is the largest entry in its block there corresponds a fall step followed by q pieces of fall steps, q being the size of each block. For the inverse mapping, number the rises of the Dyck path from 1 to n going from left to right. To each fall assign the number of its matching rise (the matching rise of a fall is the first rise to the left situated on the same level). The numbers on the same descent form a block of the non-crossing partition. Under this bijection, the number of blocks of a non-crossing partition corresponds to the number of peaks of the corresponding Dyck path.

For instance, the following Figure 7.3 shows an example of this bijection.

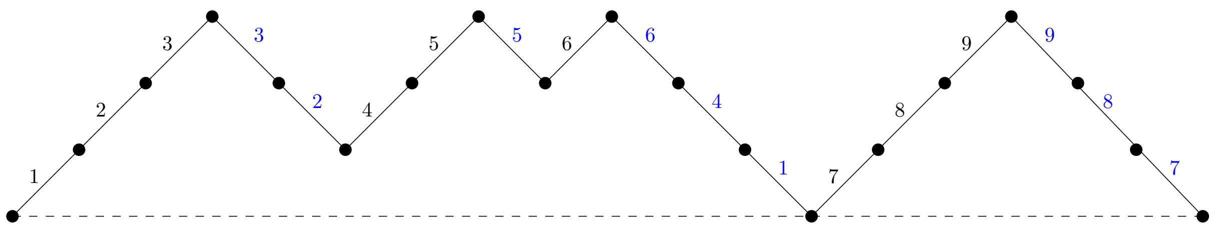

Figure 7.3: The Dyck path corresponding to the partition $\{1, 4, 6\}, \{2, 3\}, \{5\}, \{7, 8, 9\}$.

In addition, in [45] we can see the correspondance between ordered trees and non-crossing partitions where an ordered tree is a rooted tree where the order of the subtrees of a node is significant.

Given an ordered tree with n edges, we attach the numbers $1, \ldots, n$ to the edges in the following way: we traverse the tree in preorder (visit the root, then traverse its subtrees from left to right) and label an edge whenever we see it first with the smallest number not yet used. From this labelling we construct a non-crossing partition as follows: the chain, starting from the root and containing the edge labelled n forms one set. After deleting that chain, we have an ordered forest and follow the same principle. We take the longest chain containing the highest number that is yet available; this gives the second set, and so on. Now we produce the inverse map. Given a non-crossing partition, we draw a (monotonically labelled) chain using that set containing the number n. Now we take that block containing the highest number not yet used, form a chain and attach it to the first chain where it fits. If this new chain starts with number a, it fits at the node of the first chain that is between labels b, c such that $b < a < c$. If such a situation does not exist, we attach the second chain at the root of the first chain. This process will be repeated. The next chain have to lie left to the already constructed tree; there is a unique node where it fits.

An example of this bijection is given by the following Figure 7.4.

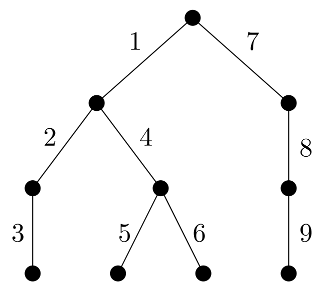

Figure 7.4: The tree corresponding to the partition $\{1,4,6\}, \{2,3\}, \{5\}, \{7,8,9\}$ with 9 edges and 4 leaves.

Conclusions

As we can verify from this thesis, geometric graph enumeration is a branch of combinatorics that presents many open problems. Throughout these pages, we have obtained some interesting results such as new production matrices for several classes of graphs based on different definition of degree, eigenvectors for each matrix and many other formulas and results related to a production matrix.

As we have said in the introduction, the work of this thesis is based in some previous results about enumeration of graphs based on production matrices. This continuation, which led to new matrices, has been possible because of the different definition of the degree we used with each matrix. While in previous papers [28, 29], the degree of the graphs was defined as the number of incident edges to the root vertex of the graph or the number of nodes on each block in the case of partitions, in this thesis, we have used another definition of degree: number of visible vertices seen from the new root vertex or the number of isolated vertices seen from the new root vertex. Different definition of degree could give us different production matrices, as we have seen in the cases of geometric graphs or non-crossing partitions. In other cases, like the one for spanning trees, diffferent definition of degree gives the same production matrix.

However, there are other important properties of the matrices on which we have been working. For instance, a complete solution to the eigenvalues of the production matrices remains elusive to us. These eigenvalues are important because the largest of them gives the assymptotic growth of the matrix and therefore, the assymptotic growth of the number of graphs of a certain class.

We dealt with this problem of finding eigenvalues in two ways. The first one was by trying to apply some iterative algorithms on the production matrices to approximate the eigenvalues. Despite the matrices being Hessenberg-Toeplitz, and there are many algorithms (Arnoldi iteration, Lanczos algorithm) that are very useful with this kind of matrices, we were not able to solve this problem. So we start by studying other ways to solve this problem. It turns out that any polynomial with degree n is the characteristic polynomial of some companion matrix of order n. Companion matrices have a very simple form, so we thought using companion matrices would be an easier method. However, the characteristic polynomial of the companion matrices was not so easy to solve, so we did not obtain any closed formula for the eigenvalues of the production matrices. We only obtained some results for the eigenvalues of the matrix of quadrangulations such as the one obtained in Theorem 10.

We also were unable to find production matrices for matchings and bipartite graphs with point sets in convex position. For instance, when we started studying matchings with the degree definition of visibility of the root node, we realised that it was quite difficult due to the fact that when we flip or remove an edge of the former graph, we could change the visibility of the current root.

Moreover, the results we have obtained, in spite of the fact that they are very interesting, are made for specific types of graphs, those with the point sets in convex position. However, any of the graph classes studied can be defined for point sets in general position, and it is here when we can find more difficulties when obtaining a production matrix for the same graph classes we have studied.

Other problems that we had no time to conclude or because of their difficulty we were not able to finish were for instance to separate the production matrix of Theorem 25 into a product of matrices without the sums in the entries of the matrix. We believe that the matrix in Theorem 25 is related to the exponential transform and Ridell's formula.

Other results that we have not finished was to find relations between the vector that counts graphs and the characteristic polynomial of a production matrix given by Cayley-Hamilton's theorem. We only found results for quadrangulations and pentangulations, such as Proposition 1 and Proposition 8.

On a personal note, I would like to conclude by pointing out how much I enjoyed working in this thesis because I was able to improve my knowledge about an interesting topic in the field of Combinatorics, and in particular graph enumeration, and also start in the world of research. Thanks to this project I had the oportunity of putting into practice what I have been studying throughout the four years of the Bachelor's Degree and this last year of Master's Degree. I would like also to thank my two advisors Clemens Huemer and Rodrigo Silveira for their patience and their help during the realization of the thesis.

Bibliography

[1] J.C. Aval, *Multivariate Fuss–Catalan numbers*, Discrete Mathematics **308** (2008) 4660−4669.

[2] C. Banderier, P. Flajolet, D. Gardy, M. Bousquet-Mélou, A. Denise, D. Gouyou-Beauchamps, *Generating functions for generating trees.* Discrete Mathematics, **246** (2002) 29−55.

[3] E. Barcucci, A.D. Lungo, E. Pergola, R. Pinzani, *ECO: a methodology for the enumeration of combinatorial objects*, J. Differ. Equations Appl. **5** (1999), 435−490.

[4] J.M. Bogoya, A. Böttcher, S.M. Grudsky, *Asymptotics of Individual Eigenvalues of a Class of Large Hessenberg Toeplitz Matrices*, Oper. Theory Adv. Appl. **220** (2012), 77−95.

[5] X.W. Chang, M.J. Gander, S. Karaa, *Asymptotic properties of the QR factorization of banded Hessenberg?Toeplitz matrices*, Numer. Linear Algebra Appl. **12** (2005), 659−682.

[6] T.S. Chow, *A class of Hessenberg matrices with known eigenvalues and inverses*, SIAM Rev. **11** (1969), 391−395.

[7] F.R.K. Chung, R.L. Graham, V.E. Hoggat, M. Kleiman, *The number of Baxter permutations*, J. Combin. Theory Ser. A **24** (1978) 382−394.

[8] S. Cleary, A. Rechnitzer, T. Wong, *Common edges in rooted trees and polygonal triangulations*, The electronic journal of combinatorics **16** (2009).

[9] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, (1974).

[10] M. Črepinšek, L. Mernik, *An efficient representation for solving Catalan number related problems*, International Journal of Pure and Applied Mathematics, **56**(4) (2009), 589−604.

[11] P. Dale, R. Vein, *Determinants and their Applications in Mathematical Physics*, Springer, New York, (1999).

[12] J.A. De Loera, J. Rambau, F. Santos, *Triangulations: Structures for Algorithms and Applications*, Algorithms and Computation in Mathematics **25** (2010).

[13] E. Deutsch, *Dyck path enumeration*, Discrete Mathematics **204** (1999), 167−202.

[14] E. Deutsch, L. Ferrari, S. Rinaldi, *Production matrices*, Adv. in Appl. Math. **34** (2005), 101−122.

[15] M. Drmota, M. Noy, A. de Mier, *Extremal statistics on noncrossing configurations*, Discrete Math. **327** (2014), 103−117.

[16] S. Dulucq, J.G. Penaud, *Cordes, arbres et permutations*, Discrete Math. **117** (1993), 89−105.

[17] G. Fairweather, *On the eigenvalues and eigenvectors of a class of Hessenberg matrices*, SIAM Rev. **13** (1971), 220−221.

[18] P. Flajolet, M. Noy, *Analytic combinatorics of non-crossing configurations*, Discrete Mathematics **204** (1999), 203−229.

[19] G.H. Golub, C.F. Van Loan, *Matrix Computations*, third ed., Johns Hopkins University Press, Baltimore, (1996).

[20] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, **347** (1990) 200.

[21] R.M. Gray, *Toeplitz and Circulant Matrices: A Review*, Foundation and Trends in Communications and Information Theory, **2**(3) (2006), 155−239.

[22] A. Greenbaum, *Iterative Methods for Solving Linear Systems*, SIAM Publications, Philadelphia, PA, (1997).

[23] W.R. Hamilton, *Lectures on Quaternions*, Dublin, (1853).

[24] M.C. Hernando, F. Hurtado, A. Márquez, M. Mora, M. Noy, *Geometric tree graphs of points in convex position*, Discrete Applied Mathematics **93** (1999), 51−66.

[25] V.E. Hoggatt, D.A. Lind, *Fibonacci and Binomial Properties of Weighted Compositions*, Journal of Combinatorial Theory, **4** (1968), 121−124.

[26] R.A. Horn, C.A. Johnson, *Matrix Analysis*, Cambridge University Press (1985), 176−180.

[27] C. Huemer, A. de Mier, *Lower bounds on the maximum number of noncrossing acyclic graphs*. European Journal of Combinatorics, **48** (2015), 48−62.

[28] C. Huemer, A. Pilz, C. Seara, R. I. Silveira, *Characteristic polynomials of production matrices for geometric graphs*, Electr. Notes Discrete Math., **61** (2017), 631−637.

[29] C. Huemer, A. Pilz, C. Seara, R.I. Silveira, *Production matrices for geometric graphs*, Electronic Notes in Discrete Mathematics **54** (2016), 301−306.

[30] C. Huemer, A. Pilz, R. I. Silveira, *A New Lower Bound on the Maximum Number of Plane Graphs using Production Matrices*, Abstracts 34th European Workshop on Computational Geometry, (2018).

[31] F. Hurtado, M. Noy, *Graph of triangulations of a convex polygon and tree of triangulations*, Comput. Geom. **13** (1999), 179−188.

[32] A. Inselberg, *On determinants of Toeplitz-Hessenberg matrices arising in power series*, J. Math. Anal. Appl. **63** (1978), 347−353.

[33] R. Johnson, *Fibonacci numbers and matrices*, (2009). http://maths.dur.ac.uk/ dma0rcj/PED/fib.pdf.

[34] G. Kreweras, *Sur les partitions non croisres d'un cycle*, Discrete Math. **1** (1972), 333−350.

[35] T. Koshy, Thomas, *Catalan Numbers with Applications*, Oxford University Press (2008)

[36] A. Luzón, D. Merlini, M.A. Morón, R. Sprugnoli, *Identities induced by Riordan arrays*, Linear Algebra and its Applications, **436** (2012), 631−647.

[37] P. Marketos, T.C. Scott, *On the origin of the Fibonacci Sequence*, MacTutor History of Mathematics archive, University of St Andrews, (2014).

[38] D. Merlini, M.C. Verri, *Generating trees and proper Riordan arrays*, Discrete Mathematics **218** (2000), 167−183.

[39] A. Nakamoto, Y. Suzuki, *Diagonal Slides and Diagonal Rotations in Quadrangulations on the Sphere*, Yokohama Mathematical Journal **55** (2009), 105−112.

[40] M. Noy, *Enumeration of non-crossing trees on a circle*, Discrete Math. **180** (1998), 301−313.

[41] Oeis.org. The On-Line Encyclopedia of Integer Sequences®(OEIS®). [online] Available at: https://oeis.org/?language=english

[42] A. Panholzer, H. Prodinger, *Bijections for ternary trees and non-crossing trees*, Discrete Math. **250** (2002), 181−195.

[43] B.N. Parlet, *The symmetric eigenvalue problem*, SIAM, Classics in Applied Mathematics (1998).

[44] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Math. **68** (1937), 145−254.

[45] H. Prodinger, *A correspondance between ordered trees and noncrossing partitions*, Discrete Math. **46** (1983), 205−206.

[46] J.H. Redfield, *The theory of group-reduced distributions*, American J. Math. **49** (1927), 433−455.

[47] A. Regev, *The central component of a triangulation*, Journal of Integer Sequences, **16** (2013).

[48] D.G. Rogers, *Pascal triangles, Catalan numbers and renewal arrays*, Discrete Math. **22** (1978), 301−310.

[49] A. Schuetz, G. Whieldon, *Polygonal Dissections and Reversions of Series*, Involve, **9** (2016), 223−236, arXiv:1401.7194.

[50] L.W. Shapiro, *A Catalan triangle*, Discrete Mathematics, **14** (1976), 83−90.

[51] D.D. Sleator, R.E. Tarjan, W.P. Thurston, *Rotation distance, triangulations, and hyperbolic geometry*, Proc. 18th Annual ACM Symposium on Theory of Computing, (1986), 122−135.

[52] R.Sprugnoli, *Riordan arrays and combinatorial sums*, Discrete Math. **132** (1994), 267−290.

[53] R.P. Stanley, *Catalan numbers*, Cambridge University Press (2015).

[54] R.P. Stanley, *Enumerative combinatorics* **2**, Cambridge Studies in Advanced Mathematics **62** Cambridge University Press (1999).

[55] R.P. Stanley, S. Fomin, *Enumerative Combinatorics*, **2**, Cambridge University Press (2001).

[56] D. Stanton, D. White, *Constructive Combinatorics*, Springer, Berlin, (1986).

[57] P. Tilli, *Singular Values and Eigenvalues of Non-Hermitian Block Toeplitz Matrices*, Linear Algebr and its Applications **272** (1998), 59−89.

[58] L. Turyn, *Advanced Engineering Mathematics*, CRC Press (2013).

[59] E.E. Tyrtyshnikov, *A Unifying Approach to Some Old and New Theorems on Distribution and Clustering*, Linear Algebr and its Applications **232** (1996), 1−43.

[60] J. West, *Generating trees and the Catalan and Schröder numbers*, Discrete Math. **146** (1995), 247−262.

[61] J. West, *Generating trees and forbidden subsequences*, Discrete Math. **157** (1996), 363−374.

[62] J.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford University Press, Oxford, (1965).

[63] H. Xu, *Functions of a matrix and Krylov matrices*, Linear Algebra and its Applications **434** (2011), 185−200.